Provable Optimization of Quantal Response Leader-Follower Games with Exponentially Large Action Spaces

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ABSTRACT

Leader-follower games involve a leader committing strategies before her followers. We consider quantal response leader-follower games, where the followers’ response is probabilistic due to their bounded rationality. Moreover, both the leader’s and followers’ action spaces are exponentially large with respect to the problem size, hence rendering the overall complexity to solve these games beyond NP-complete. We propose the XOR-Game algorithm, which converges in linear speed towards the equilibrium of convex quantal response leader-follower games (#P-hard to find the equilibrium even though convex). XOR-Game combines stochastic gradient descent with XOR-sampling, a provable sampling approach which transforms highly intractable probabilistic inference into queries to NP oracles. We tested XOR-Game on zero-sum and distribution matching leader-follower games. Experiments show XOR-Game converges faster to a good leader’s strategy compared to several baselines. In particular, XOR-Game helps to find the optimal reward allocations for the Avicaching game in the citizen science domain, which harnesses rewards to motivate bird watchers towards tasks of high scientific value.

KEYWORDS

Leader-follower games; Quantal response; XOR-Game

1 INTRODUCTION

Leader-follower games, also known as the Stackelberg games [13], involve leaders committing strategies before her followers. Over the years, leader-follower games have been studied extensively with their wide applications in security [44, 53], crowdsourcing [54, 55], AI for social good [19, 39], etc. Recent studies have focused on the participants’ bounded rationality [43]. In other words, players do not always play the best moves due to imperfect information or limited computational capacity.

In quantal response leader-follower games, the followers take probabilistic actions due to their bounded rationality. In a Logistic quantal response game, a follower maximizes her utility, albeit together with extreme value distributed latent factors. In the eyes of an observer who does not know the latent factors, the follower’s behavior is an exponential family distribution. Quantal response games have been studied in security games [11], and in games for social good [36]. The authors of [31] propose an iterative approach to compute a near-optimal strategy for the leader in response to quantal responding adversaries.

In this paper, we consider solving quantal response leader-follower games, in which the sizes of the action spaces of both the leader and the followers grow exponentially quickly w.r.t. the problem size. Exponentially large action spaces are prevalent in real-world games, e.g., in real-time strategy (RTS) games [6] or security games [11]. They pose significant challenges in finding the equilibrium of the game because they prevent the leader from enumerating the entire action space, letting along reasoning about the followers’ responses. Although several algorithms have been proposed in computing the equilibrium for normal form games [33, 40] and extensive form games [8, 48, 51], their mathematical programs involve summing over all the followers’ actions. This becomes intractable as the number of actions grows exponentially in the size of the game.

We propose XOR-Game, the first algorithm which converges in linear speed towards the equilibrium of a convex quantal response leader-follower game with exponentially large action spaces. Despite the game is convex with respect to the leader’s strategy, the problem is still at least #P-hard due to the inference of the followers’ actions from exponentially many probabilistic choices. Overall XOR-Game optimizes for the leader’s objective following a Stochastic Gradient Descent (SGD) process. Our innovation is to harness XOR-sampling in the estimation of the gradient direction of each SGD step towards the optimal leader’s strategy. XOR-sampling transforms the highly intractable (#P complete) probabilistic inference and sampling problems into queries to NP oracles using randomly generated XOR constraints. In another view, our XOR-Game algorithm transforms the highly intractable problem of reasoning about the followers’ probabilistic actions into problems within the NP complexity class while obtaining provable guarantees on the linear convergence speed and the distances towards...
the optimum. Notice other sampling approaches, e.g., MCMC sampling, provide unbiased samples only after an exponential number of burn-in steps. This is impossible in practice, and hence using these sampling approaches cannot result in similar convergence bounds as ours. Our guarantee is also significantly stronger than those offered by e.g., variational approaches [4, 24, 26–28, 42, 52], which are typically lower bounded only and can be arbitrarily loose. Even though the idea of incorporating XOR-sampling appears straightforward, all the theoretical derivation towards linear convergence guarantees cannot borrow from existing theoretical results of SGD. The key difficulty is due to that XOR-sampling only provides constant approximation guarantees for the probability of drawing samples but cannot guarantee unbiased sample estimation, which unfortunately was needed by most prior analysis.

Among many real-world applications, we consider two special cases of convex quantal response leader-follower games. The first is a zero-sum game, where the leader is to minimize the expected utility of the followers. The second is a distribution-matching leader-follower game, which the leader harnesses rewards to move the distribution of the followers’ actions towards a given distribution. Our games have applications in the citizen science domain, where the organizer harnesses rewards to motivate citizen scientists towards tasks with high scientific value. In particular, we apply our game in the recently deployed Avicaching game [54, 55] in the eBird citizen science framework, where the organizer encourages bird enthusiasts towards bird watching activities in remote and under-sampled locations. The experiment evaluations on both synthetic games as well as on real-world Avicaching games show that our XOR-Game is able to produce better leader’s strategies in fewer SGD iterations compared to competing approaches.

2 PRELIMINARIES
2.1 Quantal Response Model
Classic decision theory takes the rational agent assumption, in which agents make perfect choices to maximize their utilities. However, this assumption falls short in the explanation of probabilistic decision-making and occasional deviation from optimal choices. Random utility models were developed to capture the bounded rationality of human decisions [5, 50]. Since its inception, random utility models have been used extensively in modeling human decision-making, ranging from demand prediction [3, 5, 50], behavior modeling [23, 25] to crowd-sourcing [17, 45].

Quantal response games, originated from Quantal choice model [32], were developed from random utility behavior models, and have achieved promising results modeling the bounded rationality of human beings [12, 37]. When faced with \( N = 2^n \) choices, where the \( i \)-th choice has an observable utility value \( V_i \), quantal response model assumes that the agent’s choice \( a \) is to maximize the sum of the utility \( V_i \) and a latent factor \( \epsilon_i \):

\[
\arg\max_{a} \sum_{i=1}^{N} V_i + \epsilon_i.
\]

\( \epsilon_i \) is i.i.d. distributed in the standard Gumbel extreme value distribution Gumbel(0, 1). In other words, the probabilistic nature of the agent’s decision-making is due to the joint optimization of \( V_i + \epsilon_i \) rather than \( V_i \) only. Noted that \( \epsilon_i \) is only available to the agent but hidden from the observer. Gumbel noise is well accepted in literature to account for the stochasticity and/or irrationality of human decision-making [34, 45, 46]. Other types of noises, such as Gaussian noise, lead to other interesting models, such as the probit model. We leave as future work to consider those models. Finally, Gumbel(0,1) is used to simplify the theoretical derivation of our algorithm. Gumbel distributions with other parameters can be considered in a similar way.

Under the random utility model, it can be proven that in the eyes of an observer who do not have access to \( \epsilon_i \), the probability that the agent chooses the \( i \)-th option is given by the following exponential family distribution:

\[
P(i) = \frac{\exp(V_i)}{Z} = \frac{\exp(V_i)}{\sum_{j=1}^{N} \exp(V_j)}.
\]

Here, \( Z = \sum_{j=1}^{N=2^n} \exp(V_j) \) is known as the partition function. Various quantities have been calculated for random utility behavior models, including the following expected utility:

\[
\text{Theorem 2.1.} \ [29] \ Under the random utility model, an agent’s expected utility when following the decisions made based on Equation 1 is \( \log(\sum_{j=1}^{N} e^{V_j}) + \gamma \), \( \gamma \) is the Euler-Mascheroni constant.
\]

**Exponential Action Spaces.** In this paper, we consider games with exponential many choices. In other words, \( N = O(2^n) \) and \( n \) denotes the input problem size. For example, in the Avicaching game in the citizen science domain where rewards are used to motivate crowdsourcing agents to explore sites with high scientific values, each agent’s choice is represented as a set of locations that the agent explores. Suppose there are \( n \) locations, the set of choices are all the sets of locations, the size of which is \( 2^n \).

2.2 Quantal Response Leader-Follower Games
Leader-follower games, also known as Stackelberg games, have attracted much research attention. In a game, the leader commits a strategy before her followers, often resulting in different equilibrium solutions from the Nash equilibrium where both sides commit strategies at the same time.

The random utility behavior model discussed in the previous section leads to quantal response leader-follower games. In a quantal response leader-follower game, the follower’s decision-making follows a random utility model. The equilibrium of a quantal response leader-follower game can be computed through:

\[
\min_{r} L(\{V_i, P(i)\}, r),
\]

subject to \( a = \arg\max_{i} V_i(r) + \epsilon_i, \)

\[
\epsilon_i \overset{\text{i.i.d.}}{\sim} \text{Gumbel}(0, 1), \quad \forall i \in \{1, \ldots, N = 2^n\}.
\]

Here, the leader’s objective is to minimize \( L \). The follower’s utility function for choice \( i \) has part \( V_i \) observable by the leader and depends on the leader’s strategy \( r \). It also contains an extreme value distributed latent part \( \epsilon_i \) which is only available to the follower but hidden to the leader.

Compared with the standard leader-follower game, the the quantal response game brings in the probabilistic responses of the followers into consideration, which fits better to reality in many occasions. Due to the latent factor \( \epsilon_i \), the follower’s action is probabilistic in the eye of the leader; namely, the follower takes the \( i \)-th action
with probability \( P(i) \), which has the exponential family form in Equation 2. The objective is written as \( L(V_i, P(i), r) \) showing that the leader’s objective function can be dependent on his strategy \( r \), the follower’s observable utility \( V_i \) and/or the probabilities of making each decision \( P(i) \). The formulation in Equation 3 can be used to model both cases where the leader takes pure or mixed strategies. In the pure strategy case, the leader’s action \( r \) can be an indicator variable of which action to take. In the mixed strategy case, \( r \) becomes a vector listing the probability of taking each action. Since we assume the follower acts according to a quantal response model, she always plays probabilistic (hence mixed) strategies in the eyes of the leader.

2.2.1 Zero-sum Games. While the XOR-Game algorithm can probably optimize many quantal response leader-follower games, we mainly consider two variants for this paper. The first variant we consider is the zero sum case, where the leader is to minimize the expected utility of the follower. In other words, the leader’s objective is (according to Theorem 2.1):

\[
L_0(V_i, P(i), r) = \log \left( \sum_{i=1}^{2^n} e^{V_i(r)} \right) + Y. \tag{4}
\]

We consider a special case where \( V_i \) is linear in \( r \), namely, \( V_i(r) = \theta_i^T r + \phi_i \). Here \( \phi_i \) measures the influence of rewards on the leader’s action. \( \phi_i \) represents the intrinsic utility in choosing action \( i \) and can vary across actions (even though it does not depend on \( r \)). We show that the game in this case is convex in \( r \) (proof is in the supplementary materials):

**Theorem 2.2.** When \( V_i(r) = \theta_i^T r + \phi_i \), the zero-sum quantal response leader-follower game is convex in \( r \). Moreover, the gradient \( \nabla L_0(r) \) has the following form of an expectation:

\[
\nabla L_0(r) = \mathbb{E}_{P(i)}[\theta_i] = \sum_{i=1}^{N=2^n} P(i)\theta_i. \tag{5}
\]

It is known in a zero-sum matrix game, the Stackelberg equilibrium matches exactly to the Nash equilibrium [49, 58]. Nevertheless, we would like to point out that in our definition of a quantal response game, the follower’s action is designed to maximize her utility function \( V_i \) in addition to an unobserved \( e_i \) (Equation 1), where \( V_i \) depends on the complete information of the leader’s strategy \( r \). This definition implicitly assumes the leader commits her strategy before the follower. The Nash equilibrium, in this setting, can be difficult to be properly defined.

2.2.2 Distribution Matching Games. Another variant we consider in this paper is the game where the leader would like to stimulate certain behaviors from the follower. In particular, the leader would like to match the probability distribution of the follower’s actions \( P \) to a desired distribution \( Q \). The difference between two distributions is measured by Kullback–Leibler (KL) divergence. In other words, the leader’s objective is:

\[
L_{DM}(V_i, P(i), r) = KL(Q||P) = \sum_{i=1}^{N=2^n} Q(i) \log \frac{Q(i)}{P(i)}. \tag{6}
\]

This formulation is specially useful for mechanism design problems, e.g. in [20, 38, 41]. In cases where certain follower’s actions increase the social welfare, the leader would set high \( Q \) values to promote these actions. Again, we focus on the special case where \( V_i \) is linear in \( r \) in this paper:

**Theorem 2.3.** When \( V_i(r) = \theta_i^T r + \phi_i \) is linear in \( r \), the distribution matching leader-follower game is convex in \( r \) and the gradient \( \nabla L_{DM}(r) \) can be represented as:

\[
\nabla L_{DM} = \mathbb{E}_{P(i)}[\theta_i] - \mathbb{E}_{Q(i)}[\theta_i] = \sum_{i=1}^{N=2^n} P(i)\theta_i - \sum_{i=1}^{N=2^n} Q(i)\theta_i. \tag{7}
\]

This theorem’s proof is in the supplementary materials.

**Avicaching Game in Citizen Science.** As a specific example, we consider a leader-follower game in the citizen science domain, where the leader (citizen science game organizer) harnesses limited rewards to encourage citizen scientists (followers) to conduct observations in remote and undersampled locations. We look into the Avicaching game hosted in the eBird citizen science program [54], where rewards are used to encourage bird watchers (citizen scientists, or followers) to visit undersampled Avicaching sites, which have large scientific values but are inconvenient and/or less interesting to visit than traditional hotspots. Each bird watcher’s choice is characterized by a set of locations \( L \subseteq \{1, \ldots, l_n\} \), which represents the set of spots one plan to visit during a bird watching trip. The organizer, in this case, harnesses reward \( r = (r_1, \ldots, r_n) \) to stimulate visits to under-sampled locations. Here, \( r_i \) is the reward that a bird watcher receives when he visits location \( l_i \). The bird watchers’ (followers’) utility \( V_0(r) \) models both the intrinsic utilities to visit the location set \( L \) as well as the reward received.

2.3 XOR Sampling

Sampling from a combinatorial space has a formal complexity of \#P-complete, the difficulty of which is beyond NP-completeness. Luckily, the recently proposed XOR-Sampling algorithm, as the result of a rich line of works using streamlining randomized constraints [1, 2, 9, 15, 16, 21, 22], provides a constant approximation guarantee on the probabilities of the samples generated. XOR-sampling transforms the highly intractable sampling problem into queries to NP-oracles while obtaining provable guarantees.

The high-level idea of XOR-sampling is to harness randomized constraints to guarantee the randomness of the samples generated. Consider a simple case where \( w(x) : \{0, 1\}^n \rightarrow \{0, 1\} \) is a binary function and one would like to obtain a sample from the solution space \( \{ x : w(x) = 1 \} \) uniformly at random. Querying a NP oracle returns one \( x \) satisfying \( w(x) = 1 \) albeit not at random. XOR-sampling works by querying NP oracles to find \( x \) which satisfies \( w(x) = 1 \) and subject to a few randomized XOR constraints. It can be proven that each additional XOR constraint removes approximately half of the solutions to \( w(x) = 1 \) at random. Hence, once a desirable number of XOR constraints are added and the resulting space has only one solution, it can be proven that the only solution remaining is a random one from the original space \( \{ x : w(x) = 1 \} \). In this way, XOR-sampling is able to bound the probability of obtaining each sample within a constant multiplicative factor of its ground-truth probability. For general weighted functions XOR-sampling has similar guarantees, although the sampling process becomes
more complex. Our proposed XOR-Game algorithm depends on the following approximation bounds:

**Theorem 2.4. (Ermon et al., 2013)** Let \( w \colon \{0, 1\}^n \rightarrow \mathbb{R}^+ \) be an unnormalized weight function. \( P(x) \propto w(x) \) is the normalized distribution. Then, with probability at least \( 1 - \gamma \), XOR-Sampling \((w, \delta, \gamma)\) succeeds and outputs a sample \( x_0 \) using \( O\left( -\log(1/\delta \gamma) \right) \) \( \gamma \) NP-oracle queries. Upon success, each \( x_0 \) is produced with probability \( P(x_0) \). We have

\[
\frac{1}{\delta} P(x_0) \leq P'(x_0) \leq \delta P(x_0).
\]

Moreover, let \( \phi : \{0, 1\}^n \rightarrow \mathbb{R} \) be a function mapping binary vectors to \( \mathbb{R} \). Denote \( \phi(x)^+ = \max\{\phi(x), 0\} \) and \( \phi(x)^- = \min\{\phi(x), 0\} \) as the positive and negative part of \( \phi(x) \). Then the expectation of one sampled \( \phi(x) \) satisfies,

\[
\frac{1}{\delta} \mathbb{E}_{\phi(x)}[\phi(x)^+] \leq \mathbb{E}_{\phi(x)}[\phi(x)^+] \leq \delta \mathbb{E}_{\phi(x)}[\phi(x)^+],
\]

\[
\delta \mathbb{E}_{\phi(x)}[\phi(x)^+] \leq \mathbb{E}_{\phi(x)}[\phi(x)^+] \leq \frac{1}{\delta} \mathbb{E}_{\phi(x)}[\phi(x)^+].
\]

**Algorithm 1: XOR-Game**

**Input:** \( r_0, \{\theta_i\}_{i=1}^N, \{\phi_i\}_{i=1}^N \)

**Params:** \( T, K, \delta, \gamma \)

**for** \( t = 0 \) **to** \( T \)

**while** \( k \leq K \)

\( P(i) \propto \exp(\theta_i^T r_t + \phi_i) \)

\( l' \leftarrow \text{XOR-Sampling}(P(i), \delta, \gamma); \)

**if** \( l' \neq \text{Failure} \)

\( l_k' \leftarrow l'; k \leftarrow k + 1; \)

**end**

\( g_t \leftarrow \frac{1}{K} \sum_{k=1}^K \theta_k^T r_t; \)

\( r_{t+1} = r_t - \eta g_t; \)

**return** \( r_T = \frac{1}{T} \sum_{t=1}^T r_t \)

3 XOR-GAME

The challenge in solving quantal response leader follower games is the intractable probabilistic inference over the follower’s strategies. In this paper, we consider games in which the follower’s action space is exponentially large. In other words, the number of actions \( N \) is of size \( O(2^n) \), where \( n \) is the problem size. These games are prevalent in real world. See the Avicaching game presented in the experiment section for an example. Notice we assume there are an compact representation for all \( \theta_i \)'s and \( \phi_i \)'s. Even though there are \( 2N \) vectors of these, we assume the availability of efficient functions \( \theta(i) \) and \( \phi(i) \). When given \( i \), they return \( \theta_i \) and \( \phi_i \), respectively. The length of encoding both functions \( \theta(i) \) and \( \phi(i) \) are within \( O(n) \), i.e., the length of the problem description. In this setup, the quantal response leader-follower game is at least \#P-hard, even limiting to the convex games considered in Theorem 2.2 and 2.3. This is because it is already \#P-hard to compute the partition function in \( P(i) \) in Equation 2. In other words, it is already \#P-complete to evaluate the leader’s objective function even for a fixed strategy.

We propose XOR-Game, which converges towards the equilibrium of convex quantal response leader-follower games in linear number of stochastic gradient descent iterations. The XOR-Game algorithm should enjoy the convergence bound for a wide variety of quantal response games. However, the actual algorithms and the convergence bounds slightly differ across different game setups, due to differences in estimating the derivatives and their correspondingly different approximation bounds given by XOR-sampling. In this paper, we demonstrate such convergence bounds on the aforementioned zero-sum and distribution matching quantal response leader-follower games. However, we are confident that similar guarantees generalize to many other games.

The algorithm variants for solving zero sum game and distribution matching game are shown in Algorithm 1 and Algorithm 2. The procedures of XOR-Game0 and XOR-GameDM have minimal differences. Both algorithms apply SGD to find the optimal reward \( r \) that minimizes the leader’s objective, \( r_0 \) is the initialization of the reward vector. The follower’s observable utility is \( V_f(r) = \theta_i^T r + \phi_i \). Samples generated from XOR-sampling are used to estimate the expectations in the gradient calculation (according to Equation 5 and 7). Because XOR-Sampling has a failure rate, repeated sampling is used until a desired number of samples are obtained. \( K \) samples are drawn from the behavior model of followers, and \( S \)
samples are from the targeting model. XOR-Sampling takes parameters \((\delta, \gamma)\). After the gradient estimation, \(r_{t+1}\) from the next iteration moves from \(r_t \) following the negative gradient direction. \(\eta\) is the step size of SGD. Finally after \(T\) SGD steps, the average of \(r_1, \ldots, r_T\) is returned as the output. Denote the total variance \(V a r_p(\theta_i)(\theta_t) = E_p(\theta_t)[|\theta_t|^2] - E_p(\theta_t)|\theta_t|^2\). We can show that the convergence bound for XOR-Game to solve the zero sum game is:

**Theorem 3.1.** (Convergence for zero-sum game) In a zero sum quantal response leader follower game with objective in Equation 4 and \(V_i(r) = \theta^T r + \phi_i, r^*\) attains the minimum of the leader's objective. \(\mathbb{F}\) is the output of XOR-Game starting from \(r_0 \) and running \(T\) SGD iterations. In iteration \(t\) of SGD, \(g_t\) is the estimated gradient, i.e., \(r_{t+1} = r_t - \eta g_t\). If \(\max P |Var_p(\theta_i)(\theta_t)| \leq \sigma^2, ||r_t - r^*||_2 \leq R, \eta \leq (2 - \delta^2)/(\sigma^2 \delta), \max P |E_p(\theta_t)| \leq G, \max P |E_p(\theta_t^\eta)| \leq G, \) where \(\theta^\eta = \max(\theta^0, 0)\) and \(\theta^\eta = \min(\theta^0, 0)\), then we have:

\[
E[\mathbb{F}_t - \mathbb{F}_0] \leq \frac{\eta}{2T} \left(\sigma^2 + 2\gamma^2 + 2\eta(\delta^2 - 1)G^2 + 2(\delta^2 - 1)GR\right).
\]

**Theorem 3.2.** (Match signs at every dimension) A group of vectors \(\Theta = \{\theta_1, \ldots, \theta_N\}\) matches signs at every dimension, if for any two vectors \(\theta_i, \theta_j \in \Theta, \theta_i = (\theta_{i1}, \ldots, \theta_iL)^T, \theta_j = (\theta_{j1}, \ldots, \theta_jL)^T\), for any dimension \(k \in \{1, \ldots, L\}\), we have \(\theta_{ik} \theta_{jk} \geq 0\). The provable guarantee for XOR-GameDM requires all \(\theta_t\) in the distribution matching game to match signs at every dimension. This requirement is not too stringent. As we have pointed out, distribution matching leader follower games are usually seen in mechanism design problems, where the leader searches for a strategy to maximize certain behaviors from the followers. Here, the leader's strategy \(r\) typically represents the incentives offered to the follower. \(\theta_t\) in this case becomes indicator variables whether certain incentives are earned if the follower takes action \(i\). Due to this reason, all \(\theta_t\) are non-negative, satisfying the matching signs condition. With these definitions, the convergence bound for the distribution matching leader follower game is as follows:

**Theorem 3.3.** (Convergence for distribution matching game) Suppose a distribution matching leader-follower game has the objective in Equation 6. \(V_i(r) = \theta_i^T r + \phi_i\). Denote \(r^*\) as the optimal leader's strategy. \(\mathbb{F}\) is the output of the XOR-GameDM. Suppose \(\{\theta_1, \ldots, \theta_N\}\) match signs at every dimension, \(\max P |Var_p(\theta_i)(\theta_t)| \leq \sigma^2, \max P |E_p(\theta_i)| \leq \sqrt{G}, Var_q(\theta_i(\theta_t)) \leq \sigma^2, |\theta_i(\theta_t)| \leq G, \) when \(1 < \delta < \sqrt{2}\) is used in XOR-sampling and the SGD step size \(\eta \leq (2 - \delta^2)/(\sigma^2 \delta), ||r_t - r^*||_2 \leq R \) for all \(r_1, \ldots, r_T\), we have:

\[
E[\mathbb{F}_t - \mathbb{F}_0] \leq \frac{\delta}{2T} ||r_0 - r^*||^2 + (\delta^2 - 1) \sqrt{2\gamma GR + 2\eta \left(\sigma^2 + 2\gamma^2 + \delta^2G^2\right)} + 2\eta(\delta^2 - 1)G^2/\min(K, S).
\]

To interpret this inequality, the first term on the right-hand side of inequality \(8\) scales inversely proportional to \(T\), showing a linear convergence speed towards the optimal leader's strategy \(r^*\). The second term is the product of \((\delta^2 - 1)\) and a constant (all terms in the square bracket). This term can be minimized with more accurate (yet more expensive) XOR-sampling, bringing in \(\delta\) closer to \(1\). The term in the second line scales inversely proportional to \(\min(K, S)\), which can be minimized by increasing \(K\) and \(S\); e.g., drawing more samples. In summary, the theorem still shows a linear convergence bound and two tails terms which can be minimized via better sampling. The proof of Theorem 3.3 depends on the following Theorem 3.4, which was motivated by Theorem 3 in [14]. Nevertheless, Theorem 3.3 in [14] does not apply to the case where the difference of two XOR sampling processes are used to estimate the gradient. We therefore need to come up with novel proof techniques for distribution matching games, which yields the following Theorem 3.4:

**Theorem 3.4.** Suppose function \(f : \mathbb{R}^d \to \mathbb{R}\) is \(L\)-smooth convex. \(r^* = \arg \min_{r} f(r)\). At any point \(r\), the gradient \(\nabla f(r)\) can be decomposed into \(\nabla f(r) = \nabla p(r) - \nabla q(r)\). At the \(t\)-th iteration of SGD, \(g_t = k_t - l_t\) is the estimated gradient, i.e., \(r_{t+1} = r_t - \eta g_t, k_t (\text{or } l_t)\) is the estimation of \(\nabla p(r_t)\) (or \(\nabla q(r_t)\)). \(\{k_t, l_t, \nabla p(r_t), \nabla q(r_t)\}\) match signs at every dimension. If \(Var(k_t) \leq \sigma^2, \max \{\|k_t\|_2^2, \|l_t\|_2^2, 2G^2, \|\nabla p(r_t)\|_2^2, \|\nabla q(r_t)\|_2^2\}\) and there exists \(1 < c < \sqrt{2}, s.t.

\[
\frac{1}{c} |\nabla p(r_t)|^T \leq \mathbb{E}[k_t^T] \leq c |\nabla p(r_t)|^T
\]

\[
c |\nabla q(r_t)|^T \geq \mathbb{E}[l_t^T] \geq c |\nabla q(r_t)|^T
\]

\[
\frac{1}{c} |\nabla p(r_t)|^T \geq \mathbb{E}[q_t^T] \geq c |\nabla q(r_t)|^T
\]

\[
\frac{1}{c} |\nabla q(r_t)|^T \leq \mathbb{E}[l_t^T] \leq c |\nabla q(r_t)|^T
\]

Let \(R = \max_k ||r_t - r^*||_2\), with \(n \leq \frac{2\sigma^2}{c^2} \mathbb{F}_t - \mathbb{F}_0 \leq \frac{1}{2T} \sum_{t=0}^T r_t\), we have:

\[
\frac{c}{2T} ||r_0 - r^*||_2^2 + \left(\frac{\epsilon - 1}{c}\right) \left(\sqrt{2\gamma GR + 2\eta \left(\sigma^2 + 2\gamma^2\right)} + 2\eta \left(\epsilon + \frac{1}{c}\right)\sigma^2\right).
\]

The proof of Theorem 3.3 is to apply Theorem 3.4 to the objective function of XOR-GameDM. Notice that \(L_{DM}\) is \(L\)-smooth when the total variation \(\max P |Var_p(\theta_i)(\theta_t)|\) is bounded (proved in a lemma). Those 4 constraints on the expectation of estimated gradients can be achieved by tuning parameters of XOR-Sampling.

To prove Theorem 3.4, we need the following lemmas. The proofs of these lemmas are left in the supplementary materials:

**Lemma 3.5.** Suppose \(f \) is convex. \(r^* = \arg \min_{r} f(r)\). At the \(t\)-th iteration of SGD, \(g_t = k_t - l_t\) is the estimated gradient. \(k_t, l_t, \nabla p(r_t),\)
\[ \nabla q(r_t) \text{ match signs at every dimension, and there exists } 1 < c < \sqrt{2}, \text{ s.t. } \frac{1}{c} \| \nabla p(r_t) \| \leq \mathbb{E}[k_t^2] \leq c \| \nabla p(r_t) \|, \| \nabla p(r_t) \| \leq \mathbb{E}[k_t^2] \leq \frac{1}{c^2} \| \nabla q(r_t) \| - \frac{1}{c^2} \| \nabla q(r_t) \| \leq \mathbb{E}[l_t^2] \leq c \| \nabla q(r_t) \|, c \| \nabla q(r_t) \| \leq \mathbb{E}[l_t^2] \leq \frac{1}{c^2} \| \nabla q(r_t) \| \]

we have:

\[ \langle \nabla q(r_t), \mathbb{E}[k_t] \rangle \geq \frac{1}{c} \| \mathbb{E}[k_t] \| \mathbb{E}[l_t^2], \]

\[ \langle \nabla q(r_t), \mathbb{E}[l_t] \rangle \geq \frac{1}{c} \| \mathbb{E}[l_t] \| \mathbb{E}[k_t^2], \]

\[ \langle \nabla p(r_t), \mathbb{E}[k_t] \rangle \leq c \| \mathbb{E}[k_t] \| \mathbb{E}[l_t], \]

\[ \langle \nabla p(r_t), \mathbb{E}[l_t] \rangle \leq c \| \mathbb{E}[l_t] \| \mathbb{E}[k_t]. \]

**Lemma 3.6.** Suppose all variables and pre-conditions are defined as in Theorem 3.4. In particular, \( \text{Var}(k_t) \leq \sigma^2, \text{Var}(l_t) \leq \sigma^2, \| \mathbb{E}[k_t] \| \leq G^2, \| \mathbb{E}[l_t] \| \leq G^2, \) we have:

\[ \| \mathbb{E}[k_t] \| \leq \sigma^2 + \frac{\sigma^2}{G}, \]

\[ \| \mathbb{E}[l_t] \| \leq \sigma^2 + \frac{\sigma^2}{G}. \]

**Lemma 3.7.** Suppose all variables and conditions are defined as in Theorem 3.4. We have:

\[ \langle \nabla f(r_t), r_t - r^* \rangle \leq c \mathbb{E}[k_t] - \mathbb{E}[l_t], r_t - r^* \rangle + \sqrt{2} \left( \frac{c}{c} - 1 \right) \mathbb{E}[k_t] - \mathbb{E}[l_t]. \]

**Proof.** (Formal proof of Theorem 3.4) By L-smoothness of \( f \) for the \( t \)-th iteration,

\[ f(r_{t+1}) \leq f(r_t) + \langle \nabla f(r_t), r_t - r_{t+1} \rangle + \frac{L}{2} \| r_t - r_{t+1} \|^2. \]

\[ = f(r_t) - \eta \langle \nabla p(r_t) - \nabla q(r_t), k_t - l_t \rangle + \frac{L\eta^2}{2} \| k_t - l_t \|^2. \]

\[ = f(r_t) + \frac{L\eta^2}{2} \| k_t - l_t \|^2. \]

\[ \eta \langle \nabla p(r_t), k_t \rangle - \langle \nabla q(r_t), k_t \rangle - \langle \nabla p(r_t), l_t \rangle + \langle \nabla q(r_t), l_t \rangle. \]

Take the expectation w.r.t. \( k_t \) and \( l_t \) on both sides, and notice Equations 10, 11, 12, 13 in Lemma 3.5, we have:

\[ \mathbb{E}[f(r_{t+1})] \leq \mathbb{E}[f(r_t)] - \frac{\eta}{c} \| \mathbb{E}[k_t] \|^2 + \mathbb{E}[l_t] \|^2 + 2\eta c \mathbb{E}[k_t] \mathbb{E}[l_t] + \frac{L\eta^2}{2} \| k_t - l_t \|^2. \]

Notice \( \text{Var}(k_t) = \mathbb{E}[\| k_t \|^2] - \| \mathbb{E}[k_t] \|^2, \text{Var}(l_t) = \mathbb{E}[\| l_t \|^2] - \| \mathbb{E}[l_t] \|^2, \) and \( \text{Tr}[\text{Cov}(k_t, l_t)] = \mathbb{E}[\| k_t \|^2] - \| \mathbb{E}[k_t] \|^2, \) we further rewrite the right-hand side as:

\[ \mathbb{E}[f(r_{t+1})] \leq \mathbb{E}[f(r_t)] - \frac{\eta}{c} \| \mathbb{E}[k_t] \|^2 + \mathbb{E}[l_t] \|^2 + 2\eta c \mathbb{E}[k_t] \mathbb{E}[l_t] + \frac{L\eta^2}{2} \| k_t - l_t \|^2. \]

After re-arranging the terms, the right-hand side again becomes:

\[ \mathbb{E}[f(r_{t+1})] \leq \mathbb{E}[f(r_t)] - \frac{\eta (2 - L\eta)}{2c} \mathbb{E}[k_t] - \mathbb{E}[l_t] \|^2 + \text{tail.} \]

\[ \text{tail} \leq \frac{\eta}{c} \| \text{Var}(k_t) + \text{Var}(l_t) \| - 2\eta c \mathbb{E}[\text{Cov}(k_t, l_t)] + 2\eta (c - \frac{1}{c}) \mathbb{E}[k_t] \mathbb{E}[l_t]. \]

Using \( \text{Var}(k_t) \leq \sigma^2, \text{Var}(l_t) \leq \sigma^2 \) and Lemma 3.6, we have:

\[ \text{tail} \leq 2\eta \left( \frac{c - \frac{1}{c}}{c} \right) (\sigma^2 + G^2) + 2\eta \left( \frac{1}{c} + c \right) \sigma^2. \]

For simplicity, denote the right-hand side of the previous inequality as a constant \( C_1 \). Hence, Equation 17 becomes:

\[ \mathbb{E}[f(r_{t+1})] \leq \mathbb{E}[f(r_t)] - \frac{\eta (2 - L\eta)}{2c} \mathbb{E}[k_t] - \mathbb{E}[l_t] \|^2 + C_1. \]

Using \( \eta \leq \frac{\sqrt{c}}{L\eta} \), we can further simplify this inequality to:

\[ \mathbb{E}[f(r_{t+1})] \leq \mathbb{E}[f(r_t)] - \frac{\eta c}{2} \mathbb{E}[k_t] - \mathbb{E}[l_t] \|^2 + C_1. \]

Because \( f \) is convex, \( f(r_t) \leq f(r^*) + \langle \nabla f(r_t), r_t - r^* \rangle \). Follow the previous inequality we get:

\[ \mathbb{E}[f(r_{t+1})] \leq \mathbb{E}[f(r^*)] + \langle \nabla f(r_t), r_t - r^* \rangle - \mathbb{E}[l_t] \|^2 + C_1. \]

Because of Lemma 3.7, we can further rewrite the previous inequality as:

\[ \mathbb{E}[f(r_{t+1})] \leq \mathbb{E}[f(r^*)] + \mathbb{E}[k_t] - \mathbb{E}[l_t] - r_t - r^* + \sqrt{2} \left( \frac{c}{c} - 1 \right) \mathbb{E}[k_t] - \mathbb{E}[l_t] + C_1. \]

Define \( C_2 = C_1 + \sqrt{2} \left( \frac{c}{c} - 1 \right) \mathbb{E}[k_t] - \mathbb{E}[l_t] + r_t - r^* \), we can write:

\[ \mathbb{E}[f(r_{t+1})] \leq \mathbb{E}[f(r^*)] + \mathbb{E}[k_t] - \mathbb{E}[l_t] - r_t - r^* + \frac{\eta c}{2} \mathbb{E}[k_t] - \mathbb{E}[l_t] + C_2. \]

From the second last to the last equation, we also take expectation w.r.t. \( r_t \) on both sides. The equality holds because the randomness of \( q_t \) come from the sampling step at the \( t \)-th iteration, which is independent of \( r_t \) (whose randomness come from the first \( t - 1 \) iterations). Because \( r_{t+1} = r_t - \eta q_t \), we have:

\[ \mathbb{E}[\text{Cov}(k_t, l_t)] = \mathbb{E}[\| k_t \|^2] - \| \mathbb{E}[k_t] \|^2, \text{Var}(l_t) = \mathbb{E}[\| l_t \|^2] - \| \mathbb{E}[l_t] \|^2, \text{and Tr}[\text{Cov}(k_t, l_t)] = \mathbb{E}[\| k_t \|^2] - \| \mathbb{E}[k_t] \|^2. \]
To quantify the computational complexity of XOR-Game, we prove the following theorem in the supplementary materials detailing the number of queries to NP oracles needed for XOR-Game and XOR-Game. The proof of this Theorem is again left in the supplementary materials.

**Theorem 3.8.** XOR-Game in Algorithm 1 uses \(O(\frac{1}{\sqrt{k}} T \log (1 - \frac{1}{\sqrt{k}})) \) queries to NP oracles. XOR-Game in Algorithm 2 uses \(O(\frac{1}{\sqrt{k}} T \log (1 - \frac{1}{\sqrt{k}})) \) queries to NP oracles.

### 4 EXPERIMENTS

We demonstrate empirical evidence that XOR-Game outperforms a few competing approaches in the speed and the quality of the solutions found for both the quantal response zero-sum leader-follower games and distribution-matching games. Our evaluation is conducted on a synthetic benchmark set and a behavior model learned from real-world data obtained from the Avicaching game, which promotes bird watchers to collect data in remote and undersampled locations using the so-called Avicaching points. The baseline approaches we consider are: (1) BRQR algorithm [56], which minimizes the leader’s objective in quantal response stackelberg games. Their approach is based on a full gradient descend (GD) optimizer, hence needs to go over all the follower’s actions in each iteration and is only applicable on games with small action spaces. (2) gibbs_game, which uses SGD to minimize the leader’s objective but utilizes Gibbs sampling in the estimation of the gradient direction. (3) bp_game, which uses samples generated from the marginal probabilities computed via loopy belief propagation during SGD, [30, 35, 57] and (4) cbp_game, which harnesses the recently proposed BP chain method in generating samples in SGD [18]. For the fairness of comparisons, the leader’s objective \( L_0 \) for the zero-sum game is computed using an exact model counter Ace [10]. The leader’s objective \( L_{DM} \) for the distribution matching game is the KL-divergence, which is computed using Ace and XOR-sampling. The estimated KL-divergence is close to the groundtruth due to the constant approximation guarantee of XOR-sampling and the exactness of Ace. Additional details are in the supplementary materials.

In synthetic and real-world experiments, we use the Avicaching game as the background. In the Avicaching game, the leader (Avicaching game organizer) harnesses rewards to motivate the followers (bird watchers) to visit remote and under-sampled locations. The rewards are in the form of virtual Avicaching points, which marks the participants’ contributions to science. At the end of each season of the Avicaching game, a lottery is drawn from which Avicaching participants have opportunities to win birding gears based on how many Avicaching points they have contributed. In both the synthetic and the real-world experiments, one action that one Avicaching participant can take is to visit a set of locations \( L \). In practice, we assume bird watchers only choose between locations historically documented in the eBird dataset [47] hence we have information for all the locations. We assume the probability that one Avicaching participant visit a set of locations \( L \) is given by:

\[
P(L) \propto \exp(w^T \theta^*_F r + w^T F L + L^T W L)
\]

where \( w \) is a vector of indicator variables of visited locations. \( w^T F L + L^T W L \) is represented using symbol \( \phi \) during theoretic derivation. Each column of \( F \) includes features associated with each location, such as its landscape composition, proximity to water, etc, which affects bird watchers’ intrinsic utilities in visiting these locations. \( W \) is a matrix characterizing the changing of utilities when visiting multiple locations (e.g., bird watchers typically do not prefer visiting multiple locations of the same type). \( \theta \) is a vector of indicator variables of whether visiting location set \( L \) receives each reward. \( w \) and \( \theta \) are the relative importance of rewards and location features. For distribution matching game, we assume \( Q(L) \) has the same form as \( P(L) \) although with different parameters to promote visits to remote and under-sampled locations.

**Validation on Small Games.** We first validate that XOR-Game finds close-to-optimal leader’s strategies on small sized games. Specifically, we focus on the zero-sum quantal response game. BRQR is used as the baseline for comparison. The difference in leader’s utility values between the equilibrium and the ones found by the algorithms are shown as Loss\(_{XOR} \) and Loss\(_{BRQR} \) in Table 1. Here “Size” represents the number of different location sets a follower can visit, \( T_{XOR} \) and \( T_{BRQR} \) are the running times of different algorithms respectively. We can see from the table that both XOR-Game and BRQR find close-to-optimal leader’s strategies. Initially XOR-Game takes longer to converge, but BRQR cannot scale to modest sized games as it runs out of a 3-hour time limit for a game with action space of \( \geq 2^{32} \). XOR-Game still produces near optimal solutions in this size. Further details of this experiment (in particular, the speed the two algorithms converge to these solutions) are left to the supplementary materials.

**Evaluation on Large Synthetic Benchmarks.** We further evaluate the performance of XOR-Game on both the zero-sum game and the distribution matching game on large synthetic benchmarks. In these experiments, we intentionally increase the dimensionality of the reward vector \( r \) to be quadratic in the number of locations to increase the difficulty of benchmarks. To be specific, we let \( \theta = \text{vector}(L L^T) = (l_1, l_2, \ldots, l_n, \ldots, l_{mn})^T \) where \( L \) is the location set vector \( L = (l_1, \ldots, l_n)^T \). This is to assume one participant can receive a unique reward \( r_{ij} \) by visiting location pair \( (i, j) \). We enforce each \( r_{ij} \) to be non-negative and no greater than 1. During SGD, when \( r_{ij} \) becomes negative (or bigger than 1), we reset it to be 0 (or 1). Additional details are in the supplementary materials.

Figure 1 (left and middle) shows the performance of various algorithms as the optimization progresses. Here, each curve shows the leader’s objective averaged over 20 benchmarks. For each benchmark, we let all algorithms start from the same initial solution. When computing the average, we normalize the objective function

<table>
<thead>
<tr>
<th>Size</th>
<th>Loss(_{XOR} )</th>
<th>Loss(_{BRQR} )</th>
<th>( T_{XOR} )</th>
<th>( T_{BRQR} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 2^2 )</td>
<td>0.0425</td>
<td>0.0071</td>
<td>147.59s</td>
<td>4.10s</td>
</tr>
<tr>
<td>( 2^3 )</td>
<td>0.0677</td>
<td>0.0149</td>
<td>154.96s</td>
<td>28.11s</td>
</tr>
<tr>
<td>( 2^5 )</td>
<td>0.0346</td>
<td>0.0139</td>
<td>196.48s</td>
<td>46.16s</td>
</tr>
<tr>
<td>( 2^{16} )</td>
<td>0.0338</td>
<td>0.0178</td>
<td>302.24s</td>
<td>499.57s</td>
</tr>
<tr>
<td>( 2^{32} )</td>
<td>0.0814</td>
<td>N/A</td>
<td>699.36s</td>
<td>( &gt;3h )</td>
</tr>
<tr>
<td>( 2^{64} )</td>
<td>0.0799</td>
<td>N/A</td>
<td>8476.04s</td>
<td>( &gt;3h )</td>
</tr>
</tbody>
</table>
values against that of the initial solution so each algorithm always starts from an objective function value of 1.

Notably, XOR-Game descends to the best solutions within the least number of SGD iterations for both the zero-sum games and the distribution matching games. We would like to point out that XOR-sampling in this case is efficient in obtaining the samples, even though XOR-sampling has to answer NP-complete queries. In particular, it roughly takes 1 second for XOR-sampling to obtain 100 samples during SGD, but in general it takes 50 seconds for Gibbs sampling (300 MCMC steps), 2.8 seconds for belief propagation, and 4700 seconds for chained belief propagation (cbp). Because cbp is so slow, we use 100 samples in the gradient estimation for all other approaches but only 10 samples for cbp.

**Evaluation on the Avicaching Game.** We then evaluate all approaches on a behavior model learned from real-world data collected from the Avicaching Game. The data comes from an actual field deployment of the Avicaching game in the eBird crowdsourcing platform between March 27 and October 29, 2015 (30 weeks) in Tompkins and Cortland counties of the New York State. A set of 50 Avicaching locations were selected, which were all publicly accessible but received no visits prior to the game. The goal of the Avicaching game was to shift the bird watchers’ efforts from traditional bird watching hot spots to these Avicaching locations, harnessing Avicaching points. The numbers of Avicaching points offered for each visit to these Avicaching locations were updated every Monday. The Avicaching game was remarkably effective during this field deployment. It was reported in [54] that 19% of the bird watching effort in these two counties were shifted to these Avicaching locations.

Before evaluating algorithms, we first learn a behavior model in the form of Equation 23 for all eBird participants. Because the field study gives an independent reward to each Avicaching location, we set $\theta_{L}$ to be $L, \theta_{F} = L^{T}r$, which represents the total reward from visiting the location set $L$. $F$ includes landscape features obtained from the 2011 National Land Cover Database (NLCD). $L^{T}WL$ represents the change in utility functions for visiting multiple locations. Overall, $w_f L F + L^T W L$ represents the intrinsic utility of visiting locations $L$. Each data point consists of the set of locations $L$ one bird watcher visits and the corresponding reward $r$ of the week. Parameters $w_r$, $w_f$, and $W$ are learned using Contrastive Divergence [7].

We run various algorithms for the distribution matching game to minimize the KL-divergence between the learned probability density $P(L)$ with a manually designed $Q(L)$, which promotes the visiting of under-sampled Avicaching locations and suppresses the visiting to others. The rewards were set to be greater than 0 but less than 100 for each location (same order of magnitude as the actual field deployment). Additional details in terms of learning $P(L)$ and $Q(L)$ can be found in the supplementary materials.

Figure 1(right) demonstrates that XOR-Game descends to an optimal reward allocation faster than competing approaches. All benchmarks start with identically initialized rewards. We manually inspected the solutions. The final solutions of all approaches reach almost zero for the KL-divergence, suggesting a possibility to match the learned probability density $P(L)$ to $Q(L)$ using available rewards. Nevertheless, we cannot conclude that the Avicaching game participants will act according to $Q$ if we had the opportunities to deploy the rewards into the field. This is because all calculations are based on a learned behavior model from historical data. We cannot guarantee how much the learned model captures the subtle aspects of human decision-making and new visiting patterns may emerge as the human behavior changes with the introduction of the Avicaching game. On average, the wall-clock time for each method is: XOR-game(24h), bp_game(21h), and gibbs_game(10h). The cbp_game is excluded for comparison because it takes >8h per SGD iteration. In summary, XOR-Game requires the least number of SGD iterations to descend to the best leader’s strategies among all benchmark algorithms. XOR-game completes in a reasonable time, has good empirical performance and provable guarantees.

**5 CONCLUSION**

We proposed XOR-Game to solve the convex quantal response leader-follower games with exponentially large action spaces. XOR-Game has a linear convergence speed towards the equilibrium of the leader-follower games. Our approach is based on an integration of XOR-Sampling and stochastic gradient descent, transforming the otherwise #P-hard problem into queries within the NP complexity class, while obtaining guarantees for the convergence speed. The experiments on both synthetic and real-world Avicaching games show that XOR-Game outperforms other baseline methods and hence prove its great potential for real-world applications.
ACKNOWLEDGEMENTS

We thank the anonymous reviewers for their comments and suggestions. eBird relies on the time, dedication, and support from countless individuals and organizations. We thank the many thousands of eBird participants for their contributions and the eBird team for their support. This work was funded in part by the Leon Levy Foundation, the Wolf Creek Foundation, and the National Science Foundation (ABI sustaining: DBI-1939187, IIS-1850243, CCF-1918327).

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