on weak-approximations. The currently best result about finding Brouwer weak-approximate fixed-points is the following:

Proposition 4 (Chen and Deng [11]). *An ϵ-weak approximate fixed-point of an M-Lipschitz continuous function $f : \mathbb{R}^N \to \mathbb{R}^N$ can be found through $O((1/\epsilon \cdot M)^{N-1})$ queries to f and this is tight.*

M is intuitively an upper bound on the first derivatives of f. This kind of complexity is essentially a polynomial-time approximation scheme in fixed dimension N, but exponential in N otherwise.

As we will see, sometimes the proportionality conditions we use can be described with quadratic or other types of constraints, so results from mathematical programming become relevant. A *quadratically constrained quadratic program (QCQP)* is a collection of quadratic constraints $g_i(\boldsymbol{x}) \leq 0$ and a quadratic objective function $g_0(\boldsymbol{x})$ to be minimized over all $\boldsymbol{x} \in \mathbb{R}^N$. The decision problem of the *existential theory of the reals* is to find a solution $\boldsymbol{x}$ of a formula $\varphi(\boldsymbol{x})$ that is a quantifier-free formula involving equalities and inequalities of real polynomials. The following effective theorem was proved by Renegar [26]:

Proposition 5 (Renegar [26]). *An ϵ-strong approximation $\boldsymbol{x}_\epsilon$ of a satisfying assignment $\boldsymbol{x} \in \mathbb{R}^N$ of a quantifier-free formula $\varphi(\boldsymbol{x})$ involving P polynomial inequalities of maximum degree D and with coefficients of total encoding length L can be found in time* $\max\{L, \log(1/\epsilon)\} \cdot \mathrm{polylog}(L)(PD)^{O(N)}$.

Finally, a simple heuristic to search for a fixed-point is the *simple iteration* heuristic, which constructs a sequence of points $\boldsymbol{x}^0, \boldsymbol{x}^1, \ldots$ by taking $\boldsymbol{x}^0$ an arbitrary initial point from the set K, and then setting $\boldsymbol{x}^i = f(\boldsymbol{x}^{i-1})$. This procedure converges to a fixed-point if the function f is a *contraction*, which means that there is a non-negative constant $q < 1$ such that for every $\boldsymbol{x} \in K$, $\|f(\boldsymbol{x}) - f(f(\boldsymbol{x}))\| \leq q \cdot \|\boldsymbol{x} - f(\boldsymbol{x})\|$ under some norm; this is Banach's fixed-point theorem [1]. (Essentially, the proof follows as the distance between iterations decreases geometrically with the coefficient q.) There are many other heuristics for finding fixed-points, usually in the guise of "zero-finding" or "root-finding" heuristics, because finding $\boldsymbol{x}$ such that $f(\boldsymbol{x}) = \boldsymbol{x}$ can equivalently be seen as finding a zero of the function $g(\boldsymbol{x}) = f(\boldsymbol{x}) - \boldsymbol{x}$ or $g(\boldsymbol{x}) = \|f(\boldsymbol{x}) - \boldsymbol{x}\|_1$, for example.

We are now ready to see what these results say about our four notions of proportionality.

Exact Proportionality (EP): One can verify that f is a correspondence satisfying the conditions of Kakutani's theorem, thus, a solution is always guaranteed to exist. However, because f is a correspondence and not a function, it is unclear how to define the simple iteration procedure or apply other zero-finding heuristics. The problem can be defined as a QCQP as follows.

Observation 6. *If a voter v delegates $S \in \mathcal{S}_v$ to u and $b_{v,S} > 0$, then proportionality under (EP) is equivalent to*

$$\forall c_1, c_2 \in S \quad \left(u(c_2) \neq 0 \implies \frac{v(c_1)}{v(c_2)} = \frac{u(c_1)}{u(c_2)} \right) \ .$$

From this we can derive a quadratic constraint: $v(c_1) \cdot u(c_2) = u(c_1) \cdot v(c_2)$. Notice that the constraint behaves precisely as required also in the case when u assigns zero support to S, because both sides of the equality will be zero and thus all possible values are permissible for $v(c_1)$ and $v(c_2)$. A quadratic program that models the problem consists of the following variables and constraints.

- Definition of the variables: for every voter $v \in V$ and candidate $c \in C$ we define a variable $x_{v,c} \in [0,1]$ that is the fractional support v gives to c.
- Constraint for cumulative ballots: we fix that every voter $v \in V$ splits budget 1 to the candidates,

$$\textstyle\sum_{c \in C} x_{v,c} = 1 \ . \tag{3}$$

- Delegation budget: voter $v \in V$ has to split budget $b_{v,S}$ among candidates in $S \in \mathcal{S}_v$,

$$\textstyle\sum_{c \in S} x_{v,c} = b_{v,S} \ . \tag{4}$$

- (EP) constraint for every delegation of $S \in \mathcal{S}_v$ by a voter $v \in V$ (due to Observation 6):

$$x_{v,c_1} \cdot x_{\delta(v,S),c_2} = x_{\delta(v,S),c_1} \cdot x_{v,c_2} \quad \forall c_1, c_2 \in S \ . \tag{5}$$

Thus, one can utilize the theorem of Renegar as we describe in Subsection 4.1.[3]

Exact Proportionality with Thresholds (EP-T). We have already shown in Proposition 1 that solutions might not exist. Notice that (EP-T) does not fit the conditions of Brouwer's theorem as f is not continuous. The lesson here is to be cautious when considering discontinuous best-response functions; while discontinuity does not immediately imply non-existence of solutions, it opens the door to it.

Exact Proportionality with Thresholds, Interpolated (EP-TI). Compared with (EP), f is now a function, and we can see that it satisfies the conditions of Brouwer's theorem, so a fixed-point is always guaranteed to exist. Moreover, we could now use the simple iteration heuristic, as well as any of the zero-finding heuristics. In Subsection 4.1 we show applications of Renegar's algorithm in solving (EP-TI).[4]

Weighted Convex Combinations (WCC). Finally, for (WCC), we again observe that the best-response function f is continuous and thus Brouwer's theorem guarantees the existence of a fixed-point. Moreover, f is amenable to simple iteration and other zero-finding heuristics, and it has continuous derivatives, which can be exploited by many heuristics (unlike (EP-TI), which has discontinuous derivatives). We also note that (WCC) can be modeled as a QCQP: for every voter $v \in V$ and every delegation $S \in \mathcal{S}_v$ the constraint (5) can be replaced by

$$\begin{aligned} & x_{v,c} \cdot \|\boldsymbol{d}_{v,S} + w_{v,S} \cdot \boldsymbol{x}_{\delta(v,S),S}\|_1 \\ &= b_{v,S} \cdot (d_{v,c} + w_{w,S} \cdot x_{\delta(v,S),c}) \quad \forall c \in S \ , \end{aligned} \tag{6}$$

which is quadratic in $\boldsymbol{x}$ (note that the $\| \bullet \|_1$ in the left hand side is a linear expression in terms of $\boldsymbol{x}$). This also implies that we can use the algorithm of Renegar (see Subsection 4.1).

Additionally, in [22] we describe counterexamples construction of certain algorithmically favorable structural properties for (WCC).

[3] In practice, there are also many QCQP solvers such as IPopt, Knitro, Gurobi, or Baron, that can be used to solve this problem.

[4] It is also possible to use a QCQP formulation similar to that of (EP) augmented with logical disjunctions that can be formulated using 0/1 variables, enforced as $x \cdot (1 - x) = 0$.

4.1 Parameterized Algorithms

On the positive side, since the constraints on solutions satisfying (EP), (EP-TI), and (WCC) can be formulated as logical connections of polynomial inequalities, Proposition 5 can be used to derive the following results.

Theorem 7. *We can find an ϵ-strong approximation of a solution $\boldsymbol{x} \in \mathbb{R}^{nm}$ of an instance of FGLD for CBs with n voters and m candidates satisfying any of proportionality notions (EP), (EP-TI), or (WCC) in time* $\text{polylog}(\boldsymbol{w}, \boldsymbol{d}, 1/\epsilon) \cdot (nm)^{O(nm)}$.

Proof. Our goal is to construct a formula $\varphi(\boldsymbol{x})$ describing a solution $\boldsymbol{x}$, and then apply Renegar's algorithm (Proposition 5). For (EP), consider the QCQP given by constraints (3)–(5). A formula φ expressing that $\boldsymbol{x}$ satisfies all of these constraints is simply their conjunction, the number of constraints is bounded by $O(nm)$, the largest degree is 2, and the largest coefficient is 1. For (EP-TI), we can use an implication: if $\|\boldsymbol{x}_{\delta(v,S),S}\|_1 \geq \epsilon_{v,S}$, then constraint (5) must hold, otherwise a different quadratic constraint given by f must hold. The number of constraints is again $O(nm)$, but now the encoding length of coefficients depends on the largest weight $w_{v,S}$ and default vector $\boldsymbol{d}$. This is no issue because the encoding length anyway only enters the complexity of Proposition 5 polynomially. For (WCC), simply consider the QCQP given by (3), (4), and (6). The estimates are the same as for (EP-TI). □

In particular, Theorem 7 implies a polynomial-time algorithm for instances of constant size. This may be seen as a critical limitation, however, note that an instance of FGLD for CBs may be kernelized and partitioned into *strongly connected delegation components* (SCDCs), which can then be solved separately according to their topological order where the final output can be constructed by merging the solutions of the individual SCDCs. This way, the non-polynomial dependence in the running time is with respect to the size of the largest SCDC, which can be much smaller than the original instance size *nm*. Note also that the size of the largest SCDC can be seen as a parameter that is essentially a distance from a trivial instance [5, 19], which is one with no delegation cycles, i.e., whose largest SCDC has size 1.

Let us state this result formally. The *delegation graph* has vertices $V \times C$, each vertex representing a decision of a voter on a candidate. The graph has a directed edge $(v, c_1) \to (u, c_2)$ if there exists $S \in \mathcal{S}_v$ such that $c_1, c_2 \in S$ and $\delta(v, S) = u$, encoding that a decision of voter v on candidate c_1 depends on a decision of voter u on candidate c_2. Note that there are directed edges from (v, c_1) to every (u, c_2) such that $c_2 \in S$. A *strongly connected delegation component* is a strongly connected component in the delegation graph. An SCDC can be seen as a non-trivial part of the delegation graph: put the SCDCs in their topological order, and notice that the variables $x_{v,c}$ corresponding to vertices (v, c) in an SCDC γ can be evaluated if the value is known for all variables $x_{u,c'}$ from the SCDCs that follow γ in the topological order.

Thus, we go over the SCDCs in their reverse topological order and, for each SCDC, solve the corresponding subinstance separately using Theorem 7. In this way, we gradually obtain the values for all the variables of the solution.

Let s be the number of vertices in the largest SCDC. As each subinstance corresponds to an SCDC, it can be solved in time $\text{polylog}(\boldsymbol{w}, \boldsymbol{d}, 1/\epsilon) \cdot (s)^{O(s)}$ using Theorem 7 and there are at most nm-many SCDCs, hence we obtain the following theorem.

Theorem 8. *We can find an ϵ-strong approximation of a solution $\boldsymbol{x} \in \mathbb{R}^{nm}$ of an instance of FGLD for CBs with n voters and m candidates satisfying any of (EP), (EP-TI), or (WCC) in time* $\text{polylog}(\boldsymbol{w}, \boldsymbol{d}, 1/\epsilon) \cdot s^{O(s)} \cdot \text{poly}(n, m)$, *where s is the number of vertices in largest SCDC.*

5 GENERALIZATIONS

We will now show a major strength of our treatment: it can be widely generalized. We start with stating under which conditions a solution is guaranteed to exist. The following is essentially a restatement of Brouwer's theorem.

Theorem 9. *Let n be the number of voters and m the number of candidates. For each $i \in [n]$, let $K_i \subseteq \mathbb{R}^m$ be a convex and closed set of possible votes of voter v_i, and let $K = K_1 \times K_2 \times \cdots \times K_n$. For each $i \in [n]$, let $f_i : K \to K_i$ be the best-response function of a voter v_i, that is, with respect to any $\boldsymbol{x} \in K$, if voter v_i chooses action $f_i(\boldsymbol{x})$, then their individual regret is 0.*

If each f_i is continuous, then there exists a fixed point $\boldsymbol{x} \in K$, that is, there exists, for each voter $v_i \in V$, an action $\boldsymbol{x}_{v_i}$, such that their regret is 0.

Proof. The function $f(\boldsymbol{x}) = (f_1(\boldsymbol{x}), f_2(\boldsymbol{x}), \ldots, f_n(\boldsymbol{x}))$ is continuous and the set K is convex and closed. The existence of a fixed-point follows from Proposition 2. □

Intuitively, the theorem above states that, if each voter v has a continuous best-response function f and their regret is $\|f_v(\boldsymbol{x}) - \boldsymbol{x}_v\|_1$, or equivalently, if their regret $r_v : K \to \mathbb{R}_{\geq 0}$ is continuous and they can unilaterally decrease it to 0, then a solution is always guaranteed to exist. (One implication of the equivalence is easy; the other direction follows by, given a regret function r_v, defining, for each $\boldsymbol{x} \in K$, $f_v(\boldsymbol{x})$ to be some action $\boldsymbol{x}_v$ which decreases the regret of v to 0 with respect to $\boldsymbol{x}$.) Let us outline a few settings which can be captured by Theorem 9.

(1) **Proportionality per bundle.** Each voter v can specify for each bundle S whether they require (EP-TI) or (WCC) for this delegation.
(2) **(WCC) for subcommittees.** A voter v may wish to delegate their decision to a *committee* of delegates: say that v designates k delegates $v_1, \ldots, v_k$, each with a weight $w_1, \ldots, w_k$, and the best response of v is to take $\boldsymbol{d}_{v,S} + \sum_{i=1}^{k} w_i \cdot \boldsymbol{x}_{v_i,S}$ and scale it to be of ℓ_1-norm $b_{v,S}$.
(3) **Large- vs small-scale decisions.** We have focused on the setting where the voter makes a "large-scale" decision of how support should be split among bundles of candidates, and delegates the "small-scale" decision within each bundle. Theorem 9 captures also the setting where the voter specifies support ratios within bundles (e.g., by specifying a non-negative $|S|$-dimensional vector $\boldsymbol{d}_S$ with $\|\boldsymbol{d}_S\|_1 = 1$ for each bundle S), but delegates the decision of how to split the total support among these bundles to a delegate $\delta(v)$.
(4) **Continuous confidence functions.** In (WCC), a voter expresses their confidence in a delegate through the weight

$w_{v,S}$. The influence of $\delta(v,S)$ increases with $w_{v,S}$ and decreases with $\|x_{\delta(v,S),S}\|_1$. A voter may specify a less straightforward interaction. Imagine that there is a candidate c which v is strongly in favor of, and will trust a delegate $\delta(v,S)$ to the degree to which $\delta(v,S)$ is also in favor of c. As long as the dependence of the confidence of v in $\delta(v,S)$ is continuous, satisfying solutions are guaranteed by Theorem 9.

(5) **Spatial voting.** In spatial voting [15], a voter's ballot is some real D-dimensional vector. We may define FGLD in this setting analogously to the previous point—the action of v will be a combination of their default vote and the solution of their delegate(s) to the degree of the (continuous) confidence of v in $\delta(v,S)$.

Turning to tractability, we have the following meta-theorems.

Theorem 10. *Let n, m, and, for each $i \in [n]$, K_i and f_i, be defined as in Theorem 9. Then:*

(1) *If each f_i is M-Lipschitz continuous, then an ϵ-weak approximate fixed-point $\boldsymbol{x}$ can be found through $(1/\epsilon \cdot M)^{O(nm)}$ queries to f.*
(2) *If each f_i is continuous and can be expressed by a quantifier free formula $\varphi_i(\boldsymbol{x})$ with at most P polynomials of maximum degree D and maximum coefficient encoding length L, then an ϵ-strong approximation of a solution can be found in time polynomial in L, P, D and $\log(1/\epsilon)$, if n and m are fixed constants.*

Proof. The theorem is a straightforward application of Propositions 4 and 5, respectively. □

6 DISCUSSION

We studied fine-grained liquid democracy for cumulative ballots, and concentrated on how to resolve voters' delegations transitively in a way that is proportional. In the context of fine-grained liquid democracy, our results allow for increasing voter expressiveness and flexibility and thus advance the state of the art and what is possible to do with liquid democracy. Our work does have some limitations that naturally lead to the following directions for future research.

First, we presented parameterized algorithms in Subsection 4.1 but actually we did not prove computational hardness of the problem. Indeed, ideally, one would prove, e.g., PPAD-hardness (see our comments after Proposition 3). The best evidence for hardness is a recent paper of Papadimitriou *et al.* [25], showing that there are closely-related games which finding Nash equilibrium is PPAD-hard. This seems like an intriguing but non-trivial open question.

Second, note that in our work we concentrated on how to resolve delegations, and not on how to aggregate voter preferences. Thus, we do not consider issues of social welfare directly; that is, while the increase of voter flexibility and expressiveness intuitively allow for better quality of the collective decision, a natural future direction is to complement our research with an investigation dealing with the social welfare. Such a study may follow related work such as that which was done for participatory budgeting [3]. Note that a related issue that we do not consider, for similar reasons, is that of strategic voting: this is so as, again, we are interested in the resolution of voters' delegations and not in the communal decision to be made—studying strategic voting is indeed another natural future research direction to investigate.

Third, our analysis demonstrates that the mathematics of FGLD for CBs is non-trivial, in particular as different natural notions of proportionality lead to different results and suffer from different shortcomings. Correspondingly, a natural future research direction may experiment with our proportionality notions and evaluate them in practice as well as suggest different notions of proportionality. Our work does provide some guidance towards a practical implementation of our ideas, in particular, our analysis leads us to conclude that WCC is the currently-best proportionality notion. This is because: (1) it has stronger continuity properties than other models (e.g., a continuous derivative), making it better-behaving with respect to many heuristics; and (2) our QCQP formulation can be easy to use in practice by applying quadratic programming solvers (see the discussion in Section 4).

In a more general context, we view our theoretical treatment—culminating in our meta-theorems—as an important result that could be used for other settings (such as those briefly discussed in Section 5) as well. In particular, our meta-theorems can be used in social choice settings that are continuous in nature; a particularly promising area is that of spatial voting [15].

Besides using our meta-theorems for such continuous social choice settings, an interesting avenue for future research is to develop analogous meta-theorems for discrete settings. This may be possible using fixed-point theorems for discrete functions, and the logic would be, similarly to the continuous setting, to view a given social choice setting as a game, define appropriate regret functions and apply discrete fixed-point theorems. Even if the conditions of discrete fixed-point theorems could not be satisfied, one can consider the analogue of a mixed Nash equilibrium, where a solution would not be a single action but rather a distribution on player's actions. Such meta-theorems may be used also to revisit the setting of Ordinal FGLD [8] and Knapsack FGLD [20].

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