ABSTRACT
DeGroot-style opinion formation presumes a continuous interaction among agents of a social network. Hence, it cannot handle agents external to the social network that interact only temporarily with the permanent ones. Many real-world organisations and individuals fall into such a category. For instance, a company tries to persuade as many as possible to buy its products and, due to various constraints, can only exert its influence for a limited amount of time. We propose a variant of the DeGroot model that allows an external agent to interact with the permanent ones for a preset period of time. We obtain several insights on maximising an external agent’s influence in opinion formation by analysing and simulating the variant.

KEYWORDS
Opinion Formation; DeGroot Model; Social Network

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agent’s influence. We illustrate them in the context of a company promoting its products through TV commercials.

Coverage: the number of agents to which the external agent can exert its influence. This is the expected number of viewers of the TV commercial each time it is broadcast.

Duration: the number of times the external agent can exert its influence. This is the expected number of times the TV commercial is broadcast.

Intensity: the amount of influence the external agent can exert on other agents each time it does so. This reflects the TV commercial’s impact on its viewers each time it sees it.

Timing: the time points at which the external agent exerts its influence. This reflects the time points at which the TV commercial is broadcast.

In this section, we present our opinion formation model. It involves cycles is one.

Duration factor. We also derive several other insights that deepen our understanding of opinion formation in general.

After giving some preliminaries, we present our model that incorporates the above factors into the opinion formation process. This is followed by insights obtained by analytical methods and subsequently simulations.

2 PRELIMINARIES

In this paper, we write matrices as uppercase letters in boldface such as T, vectors as lowercase letters in boldface such as p, and scalars as lowercase letters such as m. We denote the set of real numbers and integers as R and Z respectively. The entry in the i-th row and j-th column of a matrix T is denoted as t_{ij}. The transpose of a matrix T is written as T^t. A matrix is non-negative if all its entries are non-negative. A non-negative matrix T is stochastic if all its rows sum to 1, that is \sum_j t_{ij} = 1 for all i. Vectors are considered single-column matrices unless otherwise specified. The \(i\)-th component of a vector p is denoted as \(p_i\). The “zero” vector \((0, \ldots, 0)\) and the “one” vector \((1, \ldots, 1)\) are denoted as \(0\) and \(1\) respectively and are with dimensions suitable to the context they appear.

A directed graph is a pair \((V, E)\) where \(V\) is the set of nodes and \(E \subseteq V \times V\) the set of edges. In opinion formation models, a directed graph \((V, E)\) is often identified by its adjacency matrix T which is a non-negative matrix with \(t_{ij}\) representing the number of interactions between agents i and j. The matrix is stochastic if \(\sum_j t_{ij} = 1\) for all i. Paths, cycles and their lengths in a directed graph are defined in the standard way. A directed graph or equivalently an adjacency matrix is strongly connected if there is a path from any node to any other node and it is aperiodic if the greatest common divisor of the lengths of its cycles is one.

3 MODELS OF OPINION FORMATION

In this section, we present our opinion formation model. It involves permanent agents interacting temporarily with the permanent ones. Outside the external agent’s period of interaction, our model reduces to the DeGroot Model in which the pattern of interaction is represented as a \(n \times n\) stochastic matrix T. An entry \(t_{ij}\) of T represents the weight agent i places on agent j. The weights \(t_{ij}\) for \(j = 1, 2, \ldots, n\) are considered as finite resources, distributed by i to itself and others. A positive \(t_{ij}\) indicates j is able to influence i at each round of interaction; and the greater \(t_{ij}\) is, the stronger the influence. We refer to T as the interaction matrix which can be seen as the adjacency matrix that captures the social network structure. Conventionally, opinions are represented as real numbers and time is measured in rounds of interactions. We denote agent i’s opinion and the vector of the n agents’ opinions after the t-th round of interaction as \(p_i^t\) and \(p^t\) respectively. Moreover, we refer to \(p_i^0\) as the initial opinion of i and \(p^{(0)}\) the initial opinion vector. The agents repeatedly interact by synchronously taking weighted averages of the opinions of agents who can influence them, that is \(p^t\) obeys the following equation for \(t \geq 1\)

\[
p^t = T^t p^{(0)}.
\]

The weight \(t_{ij}\) is therefore the contribution of j’s opinion at each round of interaction to i’s opinion at the next round. The weight \(t_{ij}\) agent i places on itself represents its openness to other agent’s influence: \(t_{ii} = 0\) indicates an open-minded agent who totally relies on the others’ opinions whereas \(t_{ii} = 1\) indicates a stubborn agent whose opinion remains unchanged.

An opinion formation (as described by Equation (1)) is convergent if \(p^\infty = \lim_{t \to \infty} T^t p^{(0)}\) exists for any \(p^{(0)}\). A convergent opinion formation reaches a consensus if all components of \(p^\infty\) are identical, which happens when all rows of \(\lim_{t \to \infty} T^t\) are identical. We refer to \(p^\infty\) as the limiting opinion vector and \(p_i^\infty\) as the limiting opinion of i. It is shown that if the interaction matrix is strongly connected, then a convergent opinion formation always reaches a consensus. Moreover, an opinion formation with a strongly connected interaction matrix T is convergent if i is aperiodic or equivalently there is a unique left eigenvector s of T, corresponding to eigenvalue 1 such that \(\sum_{i=1}^n s_i = 1\) and

\[
p_i^\infty = \lim_{t \to \infty} T^t p_i^{(0)} = sp_i^{(0)}
\]

for \(i = 1, 2, \ldots, n\). The result establishes whether an opinion formation converges and what it converges to when it does. Also the result implies \(\lim_{t \to \infty} T^t\) is a matrix with identical rows each of which is the unique left eigenvector s. As \(p_i^\infty\) are identical for \(i = 1, 2, \ldots, n\), we refer to all of them as the limiting opinion. See the survey [24] for the other convergence and consensus conditions and [23] for the technical details on the matrix.

According to Equation (2), the limiting opinion is a weighted average of the initial opinions where agent i’s weight is \(s_i\). These weights are commonly taken as the measure of an agent’s influence in a DeGroot-style opinion formation and are sometimes referred to as the agents’ social influence [11, 18]. We refer to s as the social influence vector. Note that s being a left eigenvector of T with an eigenvalue of 1 means \(s^T s = 1\) which implies \(s_i = \sum_j t_{ij} s_j\). Thus the social influence of i is a weighted sum of the social influences of the various agents who can be influenced by i. This is a very natural property of a measure of influence and entails that an influential
person is one who is trusted by other influential persons. We adopt this measure of influence to quantify the external agent’s influence in opinion formation.

The novelty of our model lies in the treatment of the external agent that interacts with the permanent ones for a finite number of rounds. We reserve the letter $k$ for this finite number which indicates the duration of the external agent’s influence. To represent the pattern of interaction involving the external agent, we extend the interaction matrix $T$ to form a $(n+1) \times (n+1)$ interaction matrix $A$ in which the external agent is identified as the $(n+1)$th agent whose corresponding weights occupy the $(n+1)$th row and column. We refer to $A$ as the extended interaction matrix. Since the external agent acts as an organisation or individual with the solitary goal of persuading others of its opinion, the external agent is, in our terminology, a stubborn agent, hence $a_{i(n+1)(n+1)} = 1$. For the rest of the entries in the $(n+1)$th column, $a_{i(n+1)} > 0$ means the external agent is able to influence agent $i$. We reserve the letter $m$ for the number of such positive entries which indicates the coverage of the external agent’s influence. A simplifying assumption we make is that the $m > 0$ agents place an identical weight of $\lambda$ to the external agent, that is $a_{i(n+1)} = \lambda$ whenever $a_{i(n+1)} > 0$ and $i \leq n$. The weight $\lambda$ indicates the intensity of the external agent’s influence. We also assume, without loss of generality, that the $m$ agents occupy the first $m$ rows and columns of $A$. We reserve the letter $\Lambda$ for the vector that occupies the first $n$ entries of the $(n+1)$th column. Lastly, for each entry $a_{ij}$ with $1 \leq i,j \leq n$, if $a_{i(n+1)} = 0$, then it inherit the corresponding entry $t_{ij}$ in $T$, otherwise it is shrank from $t_{ij}$ by a factor of $(1 - \lambda)$ to make $A$ stochastic. Putting these together, we have

$$A = \begin{pmatrix}
(1 - \lambda) t_{11} & \cdots & (1 - \lambda) t_{1n} & \lambda \\
\vdots & \ddots & \vdots & \vdots \\
(1 - \lambda) t_{m1} & \cdots & (1 - \lambda) t_{mn} & \lambda \\
t_{(n+1)1} & \cdots & t_{(n+1)n} & 0 \\
\vdots & \ddots & \vdots & \vdots \\
t_n & \cdots & t_n & 0 \\
0 & \cdots & 0 & 1
\end{pmatrix}$$

where $m$ is the number of agents the external agent can influence, $t_{ij}$ are entries of $T$, and $\lambda$ is the weight placed on the external agent.

The $n+1$ agents interact exactly as in the DeGroot model only that it is now governed by both $T$ and $A$. The opinion vector $p^{(t)}$ obeys Equation (1) when the external agent does not participate and the following one when it does.

$$p^{(t)} = \left(A \left(p^{(t-1)} a\right)\right)_{1,...,n}$$

where $a$ is the external agent’s unchanged opinion. For a matrix $W$, we denote the matrix formed by the first $n$ rows of $W$ as $(W)_{1,...,n}$. For example, $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)_{1,2} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. To illustrate this interaction, suppose the external agent participates in the third and fourth rounds of interaction, then

$$p^{(4)} = \left(A^2 \left(T^2 p^{(0)} a\right)\right)_{1,...,n}$$

and $p^{\infty} = \lim_{t \to \infty} T^t p^{(4)}$.

Following Equation (2), after expressing the limiting opinion as the weighted average of the $n+1$ agents’ initial opinions, that is

$$p^{(\infty)}_i = \sum_{j=1}^{n+1} w_j p_i^{(0)} + \sum_{j=n+2}^{n+m+1} \frac{w_j}{\lambda} p_i^{(0)} + \sum_{j=n+m+2}^{n+1} \frac{w_j}{\lambda^2} p_i^{(0)} + \sum_{j=n+1}^{n+m+1} \frac{w_j}{\lambda^3} p_i^{(0)}$$

we take weight $w_{i(n+1)}$ as the external agent’s social influence. Due to the external agent’s intervention, the social influence vector $s$ no longer gives accurate social influence for the permanent agents. It does so if the external agent did not participate at all in which case our model reduces to the DeGroot model.

For the rest of this paper, we assume all interaction matrices are strongly connected and aperiodic to ensure an opinion formation always reaches a consensus. Also, we assume, without loss of generality, that the external agent’s fixed opinion is 1 and the permanent agents have an initial opinion of 0. Note that agents’ opinions are irrelevant to their influence and we are only concerned with the external agent’s influence. By this assumption, it follows from Equation (4) that the limiting opinion is precisely the external agent’s social influence.

### 4 ANALYTICAL RESULTS

We have argued that an organisation’s influencing effort depends on the coverage, duration, intensity, and timing of its influence. Our model captures these factors respectively as the number $m$ of agents the external agent can influence; the number $k$ of rounds of interactions the external agent participates; the weight $\lambda$ that is placed on the external agent; and the time points in which the $k$ rounds of interaction take place. Of those factors, the first three reflect the amount of resource available for the influencing effort, the more resources there are, the larger these factors’ values. As organisations usually, if not always, have a finite resource, a vital question is how and when to allocate it to achieve maximum influence.

In this section, we express the external agent’s influence as a function of the dependent factors and obtain influence-maximising insights by analysing the function. For the ease of presentation, we decompose the multiplication with the matrix $A$ in Equation (3) and rewrite it as

$$p^{(t)} = (T - \lambda (T)_m)p^{(t-1)} + \Lambda$$

Recall that we assume $a = 1$ and $\Lambda$ is the $n$-dimensional vector of which the first $m$ components are $\lambda$ and the rest are 0. For a matrix $T$, $(T)_m$ is the matrix formed by replacing all entries of $T$ with zero except those of its first $m$ rows. For example, $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)_{1,2} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

One might notice that the timing factor is intrinsically different from the other three. Apart from being unaffected by the scarcity of resources, the timing factor is not a single factor, but a collection of $k$ factors, one for each of the $k$ rounds of participation. It is impractical and pointless to consider all variations of the $k$ factors, instead, we focus on three timing options that echo real-world situations: (1) the external agent participates in $k$ rounds of interaction and it only participates after the other agents have reached a consensus; (2) the external agent participates in the first $k$ rounds of interaction; and (3) the external agent randomly participates in $k$ rounds of interaction according to a uniform distribution over a range of time points. We refer to them as consensus, start, and uniform respectively. For now, we focus on the consensus timing option. Later on, we will explain how the other two cause the explosion in the number of
variables that makes function analysis infeasible. In fact, "consensus" is the most important of the three as simulations show that it gives rise to the largest social influence.

The consensus timing option resembles the real-world situation in which an organisation always allow sufficient time for its target to thoroughly digest its influence from the previous intervention before intervening and exerting its influence again. From a function analysis perspective, it keeps the number of variables small which results in a succinct expression for the limiting opinion.

**Theorem 1.** In an opinion formation, if the external agent can influence $m \leq n$ agents and it participates in $k \geq 1$ rounds of interaction, each of which is at a time point the other agents have reached a consensus, then

$$p_i^\infty = \sum_{j=0}^{k-1} (1 - s\lambda)^j s\lambda$$

for all $i$, where $s = \sum_{m} s_j$ for the social influence vector.

**Proof.** Let $S = \lim_{T \to \infty} T^\top$, so each row of $S$ is the social influence vector $s$. We prove by induction on $k$ that $p^\infty = \sum_{j=0}^{k-1} (1 - s\lambda)^j (s\lambda, \ldots, s\lambda)^\top$ where $s = \sum_{m} s_j$. For the base case of $k = 1$. It follows from Equation (1) and (5) that

$$p^\infty = S((T - \lambda(T)m)(0 + \Lambda) = S(\lambda, \ldots, \lambda, 0, \ldots, 0)^\top = (\sum_{m} s_j \lambda, \ldots, \sum_{m} s_j \lambda)^\top = (s\lambda, \ldots, s\lambda)^\top$$

For the induction step, suppose $p^\infty = \sum_{j=0}^{k-1} (1 - s\lambda)^j (s\lambda, \ldots, s\lambda)^\top$ for $k = l$. We need to show the equality also holds for $k = l + 1$. Let $a$ be the limiting opinion vector for when the external agent participates for $l$ rounds of interaction. Due to the induction hypothesis $a = \sum_{j=0}^{l-1} (1 - s\lambda)^j (s\lambda, \ldots, s\lambda)^\top$. Then for $k = l + 1$ we have

$$p^\infty = S((T - \lambda(T)m)a + \Lambda) = S\Lambda a - \Lambda S((T)m)a + \Lambda = a - \Lambda S((T)m)a + \Lambda = a - \Lambda S(a)(0 + \Lambda) = a - \lambda S(a)(0 + \Lambda) = a - \lambda S(\sum_{m} s_j a)(0 + \Lambda)$$

$$= (1 - s\lambda)^j a + (s\lambda, \ldots, s\lambda)^\top = \sum_{j=0}^{l} (1 - s\lambda)^j (s\lambda, \ldots, s\lambda)^\top$$

This completes the induction step. Thus we have $p_i^\infty = \sum_{j=0}^{k-1} (1 - s\lambda)^j s\lambda$ for all $i$.

Essentially, the limiting opinion is the sum of the first $k$ terms of a geometric series with start term $s\lambda$ and constant ratio $1 - s\lambda$ where the scalar $s$ is the sum of the first $m$ components of the social influence vector $s$. Since our model does not specify the identities of the $m$ agents that can be influenced by the external agent, the same value of $m$ can lead to different values of $s$. So ultimately it is the value of $s$ rather than $m$ that matters in determining the limiting opinion.

We can represent the limiting opinion, which is the external agent’s social influence, as a function $I : X \times Y \times Y \to R$ such that

$I(k, \lambda, s) = \sum_{j=0}^{k-1} (1 - s\lambda)^j s\lambda$

where $X = \{x \in \mathbb{Z} \mid x \geq 1\}$ and $Y = \{y \in \mathbb{R} \mid 0 < y < 1\}$. As explained, the variable $m$ should not appear in the function and $s$ is the combined social influence of the $m$ agents for when the external agent did not participate. Substituting $I(k, \lambda, s)$ with the formula for the sum of a geometric series, we have

$I(k, \lambda, s) = 1 - (1 - s\lambda)^k$ (6)

It is easy to see that $I(k, \lambda, s)$ increases as $s$ gets larger. Since a few influential agents can have the same combined social influence as that of many less influential ones, a large value of $m$ does not necessarily give a large value of $s$. This leads to our first insight through function analysis:

(A1) if the timing option is consensus, then the external agent should aim for the more influential ones to maximise its social influence.

What this analysis also tells us is that the number of agents that can be influenced by the external agent is not the most accurate measure of “coverage.” A better choice is the combined social influence of such agents.

With the function $I(k, \lambda, s)$, we note that its three variables have nothing to do with the structure of the social network, which means the latter has no impact on the external agent’s social influence. This leads to our second insight:

(A2) if the timing option is consensus, then the structure of the social network is irrelevant to the external agent’s social influence.

The insight might seem trivial, nevertheless, it is of great significance in practice. Often a social network’s structure is unknown to an external organisation, so knowing that it is irrelevant brings certainty and assurance to an organisation’s influencing effort. Hence, we consider this irrelevance as an advantage of the "consensus" timing option over the others with which the structure does matter.

With $I(k, \lambda, s)$, we also note that the extra influence accumulated by participating one more round of interaction is $I(k + 1, \lambda, s) - I(k, \lambda, s)$. Substituting $I(k + 1, \lambda, s)$ and $I(k, \lambda, s)$ with their full expression, we have

$I(k + 1, \lambda, s) - I(k, \lambda, s) = (1 - s\lambda)^k s\lambda$. Since $0 < 1 - s\lambda < 1, I(k + 1, \lambda, s) - I(k, \lambda, s)$ decreases as $k$ gets larger. This leads to our third insight:

(A3) if the timing option is consensus, then the external agent’s influencing effort becomes less and less effective as it participates in more rounds of interaction.

Next, we will analyse $I(k, \lambda, s)$ to decide which of the three variables $s, k, \lambda$ has a more profound effect on the external agent’s social influence. As an immediate consequence of Equation (6), the following lemma shows that $I(k, \lambda, s)$ gains the same increase by scaling up $s$ as it does by scaling up $\lambda$.

**Lemma 1.** Let $r \in \mathbb{R}$, $r \geq 1$, $rs < 1$ and $r\lambda < 1$. Then

$I(k, r\lambda, rs) = I(k, \lambda, rs)$. Note that both the weight the other agents place on the external agent and the combined social influence of the agents that can be influenced by the external one are percentages, thus the preconditions in Lemma 1. The lemma leads to our four insights:
(A4) If the timing option is consensus, then it is equally effective to scale up the coverage or intensity factor to maximise the external agent’s social influence.

Since \( \lambda \) and \( s \) are equally important in determining the value of \( I(k, \lambda, s) \), it remains to compare either one of them with \( k \). The following lemma shows that \( I(k, \lambda, s) \) increases more by scaling up \( s \) than it does by scaling up \( k \).

**Lemma 2.** Let \( r \in \mathbb{Z} \), \( r \geq 2 \), \( r_s < 1 \) and \( r \lambda < 1 \). Then
\[
I(k, \lambda, rs) > I(kr, \lambda, s).
\]

**Proof.** Since, according to Equation (6), \( I(k, \lambda, rs) = 1 - (1 - r_s \lambda)^k \) and \( I(kr, \lambda, s) = 1 - (1 - s \lambda)^{kr} \), it suffices to show \( (1 - rs \lambda) < (1 - s \lambda)^{r} \). We will prove by induction on \( r \). For the base case, we have \( r = 2 \). Then \( (1 - 2s \lambda) < (1 - s \lambda)^2 \) follows from \( (1 - s \lambda)^2 = 1 - 2s \lambda + (s \lambda)^2 \) and \( s \lambda > 0 \).

For the induction step, suppose \( (1 - rs \lambda) < (1 - s \lambda)^r \) holds for \( r = l \), we need to show it also holds for \( r = l + 1 \).
\[
1 - rs \lambda = 1 - (l + 1)s \lambda
= (1 - Is \lambda) - s \lambda
\]
and
\[
(1 - s \lambda)^l = (1 - s \lambda)^{l+1}
= (1 - s \lambda)^l (1 - s \lambda)
= (1 - s \lambda)^l - s\lambda(1 - s \lambda).
\]
We have by the induction hypothesis that \( (1 - Is \lambda) < (1 - s \lambda)^l \).

Also since \( (1 - s \lambda)^l < 1 \), we have \( s\lambda > s\lambda(1 - s \lambda) \). It follows from \( (1 - Is \lambda) < (1 - s \lambda)^l \) and \( s\lambda > s\lambda(1 - s \lambda) \) that \( (1 - Is \lambda) < s\lambda < (1 - s \lambda)^l \) which implies \( 1 - rs \lambda < (1 - s \lambda)^r \). This completes the proof.

\( \square \)

Note that it is meaningless to scale up the number of participation rounds by a non-integer or by the integer 1, thus the preconditions of Lemma 2. The lemma leads to our fifth insight:

**A5.** If the timing option is “consensus,” then it is more effective to scale up the coverage or intensity factor than the duration factor to maximise the external agent’s social influence.

Finally, it is not uncommon that an organisation can influence everyone in its target group. For example, this may happen if the group is relatively small with respect to the organisation’s resources. In the remaining of this section, we deal with this special but realistic “full coverage” case.

In our model, full coverage means \( m = n \) with which Equation (5) reduces to
\[
p^{(t)} = (1 - \lambda)Tp^{(t-1)} + \Lambda
\]
where \( \Lambda \) is the \( n \)-dimensional vector \( (\lambda, \ldots, \lambda)^T \).

The distinguishing property of the full coverage case is that the timing factor does not play a part in determining the external agent’s social influence. The key to appreciating this property is the following easily verifiable equality.
\[
(1 - \lambda)T(Tp^{(t)} + \Lambda) = T((1 - \lambda)Tp^{(t)} + \Lambda)
\]

By Equation (1) and (7), if the current opinion vector is \( p^{(t)} \) then the LHS of the equality is the opinion vector after two rounds of interaction where the external agent participates in the second round and the RHS is the opinion vector after two rounds of interaction where the external agent participates in the first round. Since this “one-step” change does not affect the opinion vector after two rounds of interaction neither does it to the limiting opinion. By realising that we can repeat this “one-step” change for any number of times to have the \( k \) rounds of participation at any \( k \) time points without affecting the limiting opinion, the property must hold. This leads to our final insight obtained through function analysis.

**A6.** If the external agent can influence all agents, then the timing factor is irrelevant to its social influence.

5 SIMULATION RESULTS

Function analysis has its limits, in this section, we resort to simulations to obtain insights tied to the timing options. We think simulation might be our last resort for the start and uniform timing options. Suppose the external agent participates in the first and \( r \)th rounds where \( r \) is 2 for “start” and an arbitrary number for “uniform.” Then the limiting opinion and thus the external agent’s social influence can be derived as follows
\[
p^{\infty} = \lim_{t \to \infty} Tp^{(r)}
= \lim_{t \to \infty} T^r((T - \lambda(T)m)p^{(r-1)} + \Lambda)
= \lim_{t \to \infty} T^r((T - \lambda(T)m)T^{-2}p^{(1)} + \Lambda)
= \lim_{t \to \infty} T^r((T - \lambda(T)m)T^{-2}\Lambda + \Lambda)
= \lim_{t \to \infty} T^r(T^{-1}\Lambda - \lambda(T^{-1})m\Lambda + \Lambda)
= 2 \lim_{t \to \infty} T^r\Lambda - \lambda \lim_{t \to \infty} T^r(T^{-1})m\Lambda.
\]

Obviously, the matrix \( (T^{-1})m \) plays a part in determining \( p^{\infty} \), meaning that any variation of the social network can result in a very different social influence of the external agent. Thus, the function that expresses the social influence must consider all entries of \( T \) (as well as the \( k \) timing factors). This is an overwhelmingly large number of variables for function analysis to be feasible. For “consensus” however (\( T^{-1}m \) is cancelled out in the above derivation, making it irrelevant to determining the social influence (see the proof of Theorem 1).

In this paper, all simulations are conducted with 100 agents and with 3000 rounds of interactions. Additionally, 1000 simulations are conducted for each combination of values for the intensity, duration, coverage and timing factor. Experimenting with various simulation settings shows that a larger number of agents and simulations makes no difference to the exhibited patterns, which are already evident with as little as 10 agents and 10 simulations. Moreover, the patterns do not rely on any specific factor values, though some values make them easy to visualise.

Our first set of simulations intends to disentangle the varying effects of the timing options as the duration value grows. While holding the intensity and coverage factor constant, for each timing option, we simulate our model for duration values ranging from 0 to 45 (with an increment value of 1). In Figure 1, we plot the external
agent’s average social influence (in 1000 simulations) induced by each timing option against the duration values.

Figure 1: Comparing the timing options with respect to a growing duration factor.

Immediately we observe that the plot for “consensus” is virtually always above those of “start” and “uniform,” with a noticeable gap over the former and a tiny one over the latter. More specifically, the gap between the plots is closing towards both ends of the horizontal axis and becomes negligible at the very ends. The plots suggest “consensus” and “start” respectively gives the largest and smallest social influence while “uniform” is almost identical to “consensus.” Closing of the gap, however, cannot be interpreted as suggesting the less relevance of the timing options with small and large duration values, rather the pattern is enforced by the bounded nature of social influence values. Social influence is bounded from below by 0 and from above by 1, thus if the duration value is sufficiently small or large, then the induced social influence must be close to 0 or 1 regardless of the timing option.

Figure 2: Comparing the timing options with respect to a growing coverage factor

In a similar fashion, while holding the duration and intensity factors constant, for each timing option, we simulate our model for coverage values ranging from 0 to 0.9 (with an increment value of 0.1). In Figure 2, the average social influence induced by each timing option is plotted against the coverage values. Once again, we observe that “consensus” is the clear winner against “start” but not so much against “uniform”. Also, the gap is closing towards both ends of the horizontal axis. Likewise, the bounded nature of social influence plays a part in the closing of the gap. But this time, as the coverage value grows, the gap closes so drastically that the plots converge at a social influence (i.e., 0.8) well below 1. This means the growing coverage value also plays a part. So, unlike the duration value, it is righteous to interpret the closing of the gap as suggesting the larger the coverage value the less relevant the timing options. In fact, we have already proved a limiting case of this pattern in (A6) which concludes the irrelevance of the timing factor when the coverage value is at its largest. Hence, we obtain the following insight.

\[(S1) \text{ Once the coverage value passes certain threshold,}^{1}\text{ then the larger the coverage value, the less relevant the timing options in determining the external agent’s social influence.}\]

Figure 3: Comparing the timing options with respect to a growing intensity factor

Finally, we hold constant the duration and coverage factors and simulate our model for intensity values ranging from 0 to 0.9 (with an increment value of 0.1). The corresponding plots are given in Figure 3. Yet another time, we observe the superiority of “consensus” over “start” and “uniform,” but only slightly over the latter. This unequivocal pattern exhibited in all three sets of simulations leads to the following insight.

\[(S2) \text{ Among the three timing options, “consensus” gives rise to the largest social influence of the external agent and “start” the least.}\]

Furthermore, the fact that “consensus” and “uniform” are almost indistinguishable indicates that the key is to spread out the participation times and avoid having them in a cluster. Last but not least, we observe that, contrary to the previous simulations, the gap between the plots enlarges towards the end of the horizontal

\[1\text{The threshold depends on the intensity and duration values.}\]
axis with larger intensity values. Noticeably, whatever mechanism that drives this enlargement is very effective as it manages to do so even though the bounded nature of social influence acts in an opposite direction. This leads to our final insight.

(S3) The larger the intensity factor, the more relevant the timing options in determining the external agent’s social influence.

6 RELATED WORK

Influence maximisation is a recurring topic in studies of opinion formation and social networks. The term is often referred to as the algorithmic problem of selecting a predefined number of agents in a social network to maximise the spread of a binary opinion in an opinion diffusion process [21], [12] is the first to pose the algorithmic problem. Later on, [19] proposed the so-called independent cascade model and gave a greedy algorithm based on submodular maximisation [19, 20]. These very influential papers have since then promoted a large number of follow-up works improving and extending various aspects of the selection techniques [6]. Although the diffusion process and representation of opinions are different from ours, these works are, in our terminology, concerning the coverage factor. Rather than focusing on this single factor, we also considered the intensity, duration and timing factors and attempt to understand how the interplay between the four factors elevates and restrains the overall influencing effort.

Apart from the above works, recent years have seen a surge in the popularity of various forms of opinion diffusion in artificial intelligence [1–5, 7–9, 13, 14, 16, 26]. Some tackled influence maximisation problems from the perspective of how the sequence of opinion updates affects the influencing effort [1, 4], [1] considered the case of three alternative opinions in an asynchronous mode of opinion update and investigated the sequence of updates that maximises the spread of an opinion. [4] continued the effort and provided upper and lower bounds on the length of such sequences among other results.

7 CONCLUSION

In this paper, we generalised the DeGroot model of opinion formation to allow a temporary participant. We articulated four factors namely, duration, intensity, coverage and timing that dominate the temporary participant’s influencing effort and incorporated them into the opinion formation process. Through function analysis and simulation, we revealed the degree of importance and interplay between the factors which leads to crucial insights into influence maximisation. In summary, the temporary participant ought to adopt the consensus timing option and focus its resources on the intensity and coverage factor. The insights may aid organisations and individuals in making better strategic choices when facing limited resources to maximise their influence.

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