Game Transformations That Preserve Nash Equilibria or Best-Response Sets

Extended Abstract

Emanuel Tewolde
Foundations of Cooperative AI Lab (FOCAL), Computer Science Department, Carnegie Mellon University
Pittsburgh, USA
everancellte@cmu.edu

Vincent Conitzer
Foundations of Cooperative AI Lab (FOCAL), Computer Science Department, Carnegie Mellon University
Pittsburgh, USA
conitzer@cs.cmu.edu

ABSTRACT
In the full version of this paper, we investigate under which conditions normal-form games are (guaranteed) to be strategically equivalent. First, we show for $N$-player games ($N \geq 3$) that a) it is NP-hard to decide whether a given strategy is a best response to some strategy profile of the opponents, and b) it is co-NP-hard to decide whether two games have the same best-response sets.

We then turn our attention to equivalence-preserving game transformations. It is a widely used fact that a positive affine (linear) transformation of the utility payoffs neither changes the best-response sets nor the Nash equilibrium set. We investigate which other game transformations also possess either of the following two properties when being applied to an arbitrary $N$-player game ($N \geq 2$): (i) The Nash equilibrium set stays the same; (ii) The best-response sets stay the same.

For game transformations that operate player-wise and strategy-wise, we prove that (i) implies (ii) and that transformations with property (ii) must be positive affine. The resulting equivalence chain highlights the special status of positive affine transformations among all the transformation procedures that preserve key game-theoretic characteristics.

KEYWORDS
Strategic Equivalence; Game Transformation; Nash Equilibrium; Best Responses; Positive Affine Linear Transformation

ACM Reference Format:

INTRODUCTION
When faced with a strategic interaction with other agents, it is helpful for AI systems to detect when the current situation can be treated in the same way as another strategic game that has already been dealt with in the past. Du [13] has shown that this is generally a computationally hard task for the case of Nash equilibria. As we will show, this task is also computationally hard in the case of best responses.

Therefore, one may instead take an alternative approach for the currently encountered strategic interaction and generate a space of many other situations that share key game-theoretic characteristics, with the goal to find an instance in that space that can be analyzed and solved efficiently. More concretely, a classic tool that emerged in the beginnings of game theory has been to transform a given game into other strategically equivalent games that are easier to analyze [38]. Positive affine (linear) transformations (PATs) have been particularly useful in that regard [3, 5, 24]. To illustrate PATs, consider any 2-player normal-form game in which the players’ utilities are measured in dollars. Then, the best-response strategies of player 1 do not change if her utility payoffs are multiplied by a factor of 5. Moreover, they also do not change if 10 dollars are added to all outcomes that involve player 2 playing his, say, third strategy. More generally, PATs have the power to rescale the utility payoffs of each player and to add constant terms to the utility payoffs of a player $i$ for each strategy choice $k_{-i}$ of her opponents.

Through leveraging PATs, previous work significantly extended the applicability of efficient Nash equilibrium solvers [2, 4, 11, 37] to classes beyond those of zero-sum and rank-1 games[1] [20, 22, 28]. The key to the success of these extensions was the well-known property of PATs that they do not change the Nash equilibrium set and best-response sets when being applied to an arbitrary game. In this paper, we investigate whether there are other (efficiently computable) game transformations with that same property.

PRELIMINARIES
Normal-Form Games. Write $[n] := \{1, \ldots, n\}$ for any $n \in \mathbb{N}$. A normal-form multiplayer game $G$ specifies (a) an integer number of players $N \geq 2$, (b) a set of pure strategies $S_i = \{m_i\}$ for each player $i$ where $m_i \geq 2$ is integer, and (c) the utility payoffs for each player $i$ given as a function $u_i : S^1 \times \ldots \times S^N \rightarrow \mathbb{R}$.

We refer to the set of (strategy) profiles in $G$ as $S := S^1 \times \ldots \times S^N$. Throughout this paper (and unless explicitly specified otherwise) all considered multiplayer games shall have the same number of players $N$ and the same set of profiles $S$. Hence, any game $G$ will be determined by its utility functions $\{u_i\}_{i \in [N]}$. The players choose their strategies simultaneously, they cannot communicate with each other, and their goal is to maximize their personal utility. As usual, we allow the players to randomize over their pure strategies.

1A 2-player game, represented by its payoff matrices $A, B \in \mathbb{R}^{m \times n}$, is said to have rank 1 if $\text{rank}(A + B) = 1$. 
which extends the strategy sets to all probability distributions $\Delta(S')$ over $S'$. A tuple $s = (s^1, \ldots, s^N) \in \Delta(S^1) \times \cdots \times \Delta(S^N) := \Delta(S)$ is called a (mixed) profile in $G$. A player then optimizes for her expected utility $u_i(s) := \sum_{k \in S} s^i_k u_i(k)$.

We define $k \mapsto s^{-i}$, and $\Delta(S^{-i})$ analogously to $k, s$, and $\Delta(S)$, except that player $i$’s part is removed from it. The best-response set of player $i$ to an opponents’ profile $s^{-i} \in \Delta(S^{-i})$ is then defined as $\text{BR}_{u_i}(s^{-i}) := \arg\max_{t \in \Delta(S)} \{ u_i(t, s^{-i}) \}$, where the notation $u_i(t, s^{-i})$ stresses how player $i$ can only influence her own strategy when it comes to her payoff. A profile $s \in \Delta(S)$ is called a Nash equilibrium in $G$ if for every player $i \in [N]$ we have $s^i \in \text{BR}_{u_i}(s^{-i})$. By Nash’s result [29], any such multiplayer game $G$ admits at least one Nash equilibrium.

**Game Transformations.** Let us define the two game transformation concepts that we study in this paper.

**Definition 1.** A positive affine transformation (PAT) specifies for each player $i$ a real-valued scaling parameter $\alpha^i > 0$ and real-valued translation constants $\delta^i := (\delta^i_1, \ldots, \delta^i_N) \in \mathbb{R}^N$, for each pure profile of the opponents. The PAT $H_{\text{PAT}}$ can then take any game $G = (u_i)_{i \in [N]}$ and transform it into game $H_{\text{PAT}}(G) = (u'_{i})_{i \in [N]}$ with utility functions $u'_i : S \to \mathbb{R}, k \mapsto a_i u_i(k) + \delta^i_k$.

**Definition 2.** A separable game transformation $H$ specifies for each player $i$ a map $H^i : \mathbb{R}^N \to \mathbb{R}$, for each pure profile $k$. The transformation $H$ can then take any game $G = (u_i)_{i \in [N]}$ and transform it into game $H(G) = \{ H^i(u_i) \}_{i \in [N]}$ with utility functions $H^i(u_i) : S \to \mathbb{R}, k \mapsto h^i_k(u_i(k))$.

Separability intuitively means that the transformed game $H(G)$ has the same number of players $N$ and the same profile set $S$ as the original game $G$, and that the utility payoff of player $i$ in $H(G)$ from pure profile outcome $k$ is only a function of the utility payoff from that same player in that same pure profile outcome in $G$. We discuss in the full version of this paper why it is sensible from a computational perspective to restrict our attention to this class of game transformations. We illustrate the richness of this in the following example.

**Example 3.** Consider the $2 \times 2$ bimatrix games, that is, the case of $N = 2$ players each with $|S| = 2$ strategies. A PAT may then, for example, transform such a game $(A, B)$ into the game

$$\begin{align*}
A', B' &= \begin{pmatrix}
2a_{11} + 10 & 2a_{12} - 5 \\
2a_{21} + 10 & 2a_{22} - 5
\end{pmatrix}, \begin{pmatrix}
\frac{1}{2}b_{11} & \frac{1}{2}b_{12} \\
\frac{1}{2}b_{21} - \sqrt{3} & \frac{1}{2}b_{22} - \sqrt{3}
\end{pmatrix}.
\end{align*}$$

A separable game transformation can, for example, instead transform $(A, B)$ into the game

$$\begin{align*}
A'', B'' &= \begin{pmatrix}
-2a_{11} + 10 & a_{12}^2 \\
e^2_{22} & 0
\end{pmatrix}, \begin{pmatrix}
|h_{11}| & \text{sign}(h_{12}) \\
\sqrt{|h_{21}|} & \arctan(h_{22})
\end{pmatrix}.
\end{align*}$$

It is a well-known fact that PATs do not change the best responses (and hence, not the Nash equilibrium set, either), no matter what game $G$ they are applied to. We prove with Theorem 7 that the separable game transformation example above does not always preserve these game characteristics. In fact, each of the functions within $A''$ and $B''$ already single-handedly violates a PAT structure.

**Definition 4.** A separable game transformation $H$ (resp. map $H^i$) is said to universally preserve Nash equilibrium sets (resp. best responses) if for all games $G = (u_i)_{i \in [N]}$ the transformed game $H(G)$ has the same Nash equilibrium set as $G$ (resp. the same best-response sets as $G$, i.e., $\text{BR}_{H(u_i)}(s^{-i}) = \text{BR}_{u_i}(s^{-i})$ for all profiles $s^{-i} \in \Delta(S^{-i})$ of the opponents).

**RESULTS**

**Best Responses in Many-Player Games.** We investigate the computational complexity of problems involving best-response strategies. First, we consider the problem of deciding whether a (mixed) strategy of a player is ever a best response to some mixed profile of the opponents. This is related to rationalizable strategies [7, 30], a concept based on the idea that a rational player should eliminate any strategy that is not a best response to some belief over what her opponents may play.

**Proposition 5.** It is NP-hard to decide, given a 3-player normal-form game, whether there exist mixed strategies $r$ and $s$ of $P2$ and $P3$ such that the first pure strategy of $P1$ is a best response to $(r, s)$.

Next, we turn to best-response equivalence.

**Theorem 6.** It is co-NP-hard to decide whether two 3-player normal-form games have the same best-response sets.

**Preserving Transformations.** We give two equivalent characterizations of PATs that highlight their special status among game transformations: PATs are the only separable game transformations that always preserve the Nash equilibrium set or, respectively, the best-response sets.

**Theorem 7.** Let $H$ be a separable game transformation. Then:

- $H$ universally preserves Nash equilibrium sets for each player $i$, map $H^i$ universally preserves best responses $\iff H$ is a positive affine transformation.

The novel part about Theorem 7 is the downwards implication chain. We may circumvent this result by considering non-separable game transformations, as discussed in the full version of this paper. The space of preserving transformations may also increase if we are only interested in transforming a specific subclass of $N$-player games (provided by, e.g., domain knowledge). For a general treatment as done in this paper, it would be preferred if such a subclass still contained “most” games.

**CONCLUSION**

When faced with a strategic interaction it can be highly beneficial to consider equivalent variations of it that are easier to analyze. In the full version of this paper, we shed light on why PATs have become the go-to transformation method for that purpose, reinforcing their standing as the standard off-the-shelf approach. The current literature on game theory and on decision making in Al are lacking methods to detect or generate strategically equivalent games, and we hope that our results can serve as guidance to the development of any such detection or generation toolkit.
REFERENCES


