Selecting Representative Bodies: An Axiomatic View

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ABSTRACT
As the world’s democratic institutions are challenged by dissatisfied citizens, political scientists and computer scientists have proposed and analyzed various (innovative) methods to select representative bodies, a crucial task in every democracy. However, a unified framework to analyze and compare different selection mechanisms is largely missing. To address this gap, we advocate employing concepts and tools from computational social choice to devise a model in which different selection mechanisms can be formalized. Such a model would allow for conceptualizing and evaluating desirable representation axioms. We make the first step in this direction by proposing a unifying mathematical formulation of different selection mechanisms as well as various social-choice-inspired axioms such as proportionality and monotonicity.

KEYWORDS
(Computational) Social Choice; Liquid Democracy; Sortition; Proxy Voting; Unifying Framework; Proportional Representation

In this paper, we focus on the task of selecting a representative body, a crucial ingredient of all democratic institutions [33, 35, 37–39, 45]. There is no shortage of innovative proposals to change how representative bodies are selected around the world. For example, some propose to select representatives at random (a.k.a. sortition) [10], to elect them through transitive delegations (a.k.a. liquid democracy) [47], or to drastically increase the size of parliaments (see, e.g., https://thirty-thousand.org).

Previous research has identified various strengths and drawbacks of different selection mechanisms. However, these studies often focus on individual mechanisms, making a comprehensive and principled comparison unfeasible. Here, we present a research agenda to facilitate a structured comparative analysis and discussion.

We advocate for a principled, rigorous, and unified analysis of the strengths and weaknesses of selection mechanisms. We believe that this challenge is best approached from an axiomatic perspective.

A prerequisite for an axiomatic analysis is the formulation of selection mechanisms in a consistent framework. Thus, our envisioned research agenda consists of multiple steps. First, design a mathematical framework that can capture different selection mechanisms. Second, formulate desirable properties as axioms in this framework. Third, conduct a formal analysis of mechanisms from the perspective of these axioms. Through this process, we can build a coherent picture of the advantages and disadvantages of different proposals for selection mechanisms. As it is unlikely that a single best selection mechanism can be found, such an axiomatic approach can also be used to investigate inherent and poorly understood trade-offs at the heart of democratic innovations, for instance, by proving that certain axioms cannot be satisfied simultaneously. We believe that a joint community effort along these lines can direct public debates towards a structured discussion that would compare selection mechanisms in the context of trade-offs faced by societies, and away from arguing for and against competing selection mechanisms in an ad hoc fashion.

In the following paragraphs, we relate our agenda to computer science and briefly survey related work. Then, to make our research vision more concrete, we give an example of a simple yet rich mathematical framework, and explain how to use it to describe a mathematical framework, and explain how to use it to describe a
variety of selection mechanisms (Section 2). In Section 3, we present axioms regarding cogent representation in our framework. Finally, we conclude by detailing a range of future research challenges (Section 4).

In recent years, computer science and democratic innovations have become increasingly intertwined, with computer scientists studying different representation schemes, analyzing them axiomatically and tackling many associated algorithmic design and scalability problems. Broad adoption of complex and interactive voting methods relies on advancements in information technology [12], and Miller [40] and Tullock [46] have argued that technological advances have enabled richer political decision-making processes on a nationwide scale. In a similar spirit, our envisioned research program relies on the expertise of computer scientists. More specifically, many subcommunities of AAMAS could contribute to our endeavor to rigorously formulate and analyze selection mechanisms and desiderata. For instance, (i) the Social Choice and Cooperative Game Theory community has expertise in the axiomatic analysis of voting rules, (ii) the Coordination, Organisations, Institutions, and Norms community can contribute a normative perspective, (iii) the Humans and AI / Human-Agent Interaction community could help analyze usability aspects, and (iv) the Engineering Multiagent Systems community could help in engineering safe and verifiable tools.

In turn, improving representative selection mechanisms benefits the computer systems that make use of them. For example, some blockchains select validators via a nominated-proof-of-stake protocol, and for the security of the system it is essential that the selection is representative [14]. Further afield, Decentralized Autonomous Organizations (DAOs) are at the forefront of testing innovative governance systems based on interactive procedures [3, 32, 50].

Previous works on selection mechanisms by political and computer scientists have almost exclusively focused on analyzing the strengths and weaknesses of specific methods (e.g., [3, 8, 22, 30]), with only a few of them comparing different mechanisms. Moreover, none of these works took an axiomatic perspective; rather, they focused on epistemic aspects, the robustness of representation, and majority agreement [1, 2, 26, 28].

2 MATHEMATICAL FRAMEWORK

To make our research agenda more concrete, we outline a mathematical framework to model selection mechanisms. We say that a matrix is stochastic if each row sums up to 1. For a natural number \( n \in \mathbb{N} \), let \([n]\) denote the set \(\{1, 2, \ldots, n\}\) and let \(e_n \in \mathbb{N}^{1 \times n}\) denote the row vector containing all ones. For a vector \(a \in \mathbb{R}^{1 \times n}\), let \(\|a\|_1\) denote the \(l_1\)-norm of \(a\), i.e., \(\|a\|_1 = \sum_{i=1}^{n} |a_i|\).

2.1 Modeling Representation

We present a mathematical framework for the following task: A group \(N = [n]\) of \(n\) agents wants to select a subset of \(N\) to act as a representative body, by means of a selection mechanism \(M\). The agents selected to be part of the representative body may have different voting weights, i.e., in a decision made by the representative body, some agents’ votes can have more weight than others. Formally, given \(N\), we aim to select a weight vector \(w \in \mathbb{R}^{n\geq0}\). For each \(i \in N\), if \(w_i > 0\), then \(i\) is selected as part of the representative body and has voting weight \(w_i\). The size of the induced representative body is given by \(|\{i \in N \mid w_i > 0\}|\).

**Representation Matrix.** The relationships between the agents are captured by a representation matrix \(\Gamma \in \mathbb{R}^{n\times n}\), where the entry \(\Gamma_{ij}\) describes how well agent \(j\) can represent agent \(i\). \(\Gamma\) is a stochastic matrix that allows fractional entries, so that an agent may be represented by a mixture of other agents.\(^1\) How well agent \(i\) feels represented by agent \(j\) may be based on the issues that agent \(i\) cares about, the relative preferences of \(i\) and \(j\) on these and other issues, intrinsic characteristics of \(i\) and \(j\), and the underlying social network capturing who knows whom. The representation matrix can be interpreted as giving rise to voting behavior. Specifically, assuming that each agent can arbitrarily split their vote, the matrix entries can be considered to encode the ideal split of an agent’s vote into fractional votes. However, in this paper, as in most real-world settings, we focus on uninominal ballots, i.e., each agent can vote for exactly one other agent to be part of the representative body (though it is straightforward to extend the model to other ballot formats such as approval or ranked ballots). Thus, we interpret the entry \(\Gamma_{ij}\) as the probability that agent \(i\) selects (i.e., votes for) agent \(j\).\(^2\)

**Example 1.** Let \(N = \{A, B, C, D, E\}\) be such that \(A\) and \(B\) belong to one party, and \(C, D\) and \(E\) to another party. Suppose that \(A\) and \(E\) are extreme agents seeking power. Moreover, \(B\) and \(D\) are completely partisan and would never want to be represented by someone outside their parties. In contrast, \(C\) is moderate and could be represented by other agents with non-extreme views. This situation could be captured by the representation matrix \(\Gamma\) in Figure 1.

![Figure 1: Representation matrix \(\Gamma\) for the instance described in Example 1. Rows and columns are indexed with agents.](image)

\(^1\)\(^2\)Our model generalizes the representation model of Ebadian et al. [20] introduced in their analysis of sortition, where agents are embedded in a metric space, and distance in that space resembles representation quality.
given in Figure 1, the vector of expected vote shares of the agents is $\mathbb{E}[V] = \left( \frac{2}{3}, \frac{2}{3}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2} \right)$.

2.2 Selection Mechanisms

The next step is to formalize different selection mechanisms in our framework. To this end, we present a general model of selection mechanisms $M$ that map the agents‘ uninominal ballots to weight vectors. Our mechanisms may take two additional inputs: (1) a subset $C \subseteq N$ of agents acting as candidates with $|C| = m$, and (2) an integer $k$ specifying the target size of the representative body. For a mechanism $M$ and target $k$, we define a function $f^{M_k}$ that, given $\Gamma$ and $C$, returns the candidates’ expected voting weights under $\Gamma$ and $C$.4 Formally, $f^{M_k} : \mathbb{R}^{n \times n} \times (2^N \setminus \emptyset) \rightarrow \mathbb{R}^{m \times 1}$ is a function such that $\{i \in N \mid f^{M_k}(\Gamma, C)_{ij} > 0\}$ is a subset of $C$ of size at most $k$. Here, $f^{M_k}(\Gamma, C)_{ij}$ is the expected voting weight $\mathbb{E}[w_{ij}]$ of candidate $i \in C$ in a body of size $k$ selected by $M$, assuming that $i \in N$ votes for $j \in C$ with probability depending on $\Gamma_{ij}$.

We now describe how the expected voting weights for different selection mechanisms can be computed.

2.2.1 Direct Democracy (D). In direct democracy assemblies, all agents are included in the represented body. Thus, $C = N$ and $f^D(\Gamma, N) = \mathbf{e}_n$ for all representation matrices $\Gamma$.

2.2.2 First-Past-The-Post (F). First-past-the-post voting is widely used around the world, but it is also widely criticized for, among other things, leaving voters feeling underrepresented [9]. In first-past-the-post, the candidate receiving the highest number of votes gets a voting weight of 1; all other candidates’ voting weights are 0. Notably, in first-past-the-post elections, the electorate is typically partitioned into different voting districts, each selecting its own representative. We focus on the single-district case, but our model can be easily extended to parallel independent districts.

Continuing Example 1, let $C = \{A, B, C, E\}$. Note that the function $f^F$ alters the representation matrix to account for the set of candidates: agents can only vote for candidates, and we assume that candidates always select themselves. In the running example, $A, B, C,$ and $E$ all vote for themselves, whereas $D$ votes for $C$ and $E$ with probability 1/2 each. With probability 1/2, $C$ receives 2 votes, and $A, B$ and $E$ receive 1 vote each (so that in the representative body $C$ has a voting weight of 1 and all other agents have a voting weight of 0), and with probability 1/2, $E$ receives 2 votes, and $A, B$ and $C$ receive 1 vote each, giving $f^F(\Gamma, C) = (0, 0, 1/2, 0, 1/2)^T$.

2.2.3 Proxy Voting (P). In proxy voting, all agents are presented with a pre-defined pool of candidates, and each remaining agent can delegate their voting power to one of the candidates. All candidates are de facto members of the representative body, and candidates have a voting power proportional to the number of votes delegated to them.5 Proxy voting has been studied within both computer science [3, 16] and political sciences [40], with some works extending it to more flexible issue-based delegation [1]. The expected voting weight under proxy voting is the sum of expected delegations for the proxies (as in first-past-the-post, we adapt the representation matrix to account for the candidate set). Assuming again $C = \{A, B, C, E\}$ in Example 1, since $D$‘s vote goes to $C$ with probability 0.5 and to $E$ with probability 0.5, we get $f^P(\Gamma, C) = (1, 3/2, 0, 3/2)^T$.

2.2.4 Liquid Democracy (L). In liquid democracy, each agent can choose to be part of the representative body or delegate their vote to another agent. Delegations are transitive, i.e., if $A$ delegates to $B$ and $B$ delegates to $C$, and $C$ decides to be in the representative body, then $C$ votes on behalf of themselves, as well as $A$ and $B$. The representative body consists of all agents who self-select, with their voting power being the number of votes (transitively) delegated to them plus their own. Liquid democracy has received considerable attention in the political sciences [8, 40, 48]; computer scientists have considered it from both a procedural and an epistemic perspective [7, 13, 21, 27, 30, 31, 51], developing dedicated supporting software [5, 43], and examining possible extensions [13, 17, 24].

To find the expected voting weights of the agents under transitive delegations, we leverage the representation matrix to list all possible configurations of transitive delegations and compute their probabilities. For instance, one possible configuration in Example 1 is that every agent votes for themselves (which happens with probability $\frac{5}{14}$), resulting in all agents being part of the representative body and having voting weight 1.6 Overall, the expected voting weights are as follows: $f^L(\Gamma, N) = \left( \frac{9}{14}, \frac{22}{35}, \frac{14}{35}, \frac{1}{2}, \frac{1}{2} \right)^T$.

2.2.5 Sortition (S). Sortition is a selection method that draws $k$ agents uniformly at random from the population to act as the representative body [18, 23, 34, 42]. This allows equal access to decision-making and does not require a voting phase. All members of the representative body have equal voting weight. Thus, the expected voting weight of each agent is $\frac{1}{k}$, i.e., $f^S(k, N) = \frac{1}{k}\mathbf{e}_n$. Agents who do not participate in the representative body despite being selected pose problems with the fairness guarantees offered by sortition (self-selection creates biases, so, to protect diversity, algorithms perform stratified, not random, sampling). Computer scientists are investigating algorithmic ways to deal with this issue [22, 23].

3 AXIOMS

Presenting different selection mechanisms in a unified framework allows for their axiomatic analysis. Drawing inspiration from (computational) social choice and political sciences, we present five axioms related to different aspects of representation: proportionality, diversity, monotonicity, faithfulness, and effectiveness. Comparing selection mechanisms with respect to these axioms is a concrete research challenge that we pose; this is a first step towards a structured comparison of such mechanisms.

- Proportionality. Proportionality captures how “accurately” each candidate’s expected voting weight reflects their expected vote share. Proportionality is relevant to descriptive representation [6, 11, 36, 47, 52]; see the work of Rae [45] for an analysis of proportionality metrics for different selection formulas. We put forward

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4Some mechanisms need neither a candidate set $C$ nor the size of the representative body $k$ as input; in this case we drop $k$ from $f^{M_k}$.

5Proxy voting is closely related to the widespread practice of party-list elections [44], where agents vote for parties, and the seats in the representative body are distributed so that the number of each party’s seats is proportional to its share of received votes. We focus on proxy voting, as it allows for a cleaner mathematical formulation.

6Note that if there is a delegation cycle, the votes of agents in the cycle are lost. Accordingly, their voting weight is set to zero and the agents are effectively ignored.
We have argued that there is a need for a systematic comparison of different selection mechanisms within a unified framework, to understand the trade-offs inherent to the challenge of open democratic representation [34]. Taking a first step, we have presented a simple illustrative model that can be used to capture selection mechanisms together with representation-related axioms. We do not view our model and axioms as final or exhaustive, and we believe that asking the right questions is already the first research challenge. Nevertheless, various future research challenges within our framework will be of broad relevance.

Challenge 1: Mechanism’s Axiomatic Properties. The first challenge is to check which axioms a mechanism fulfills. For instance, we can ask: Which mechanisms are guaranteed to satisfy diversity, monotonicity, and faithfulness? Can we obtain meaningful bounds on their $\varepsilon$-proportionality or $\gamma$-effectiveness? Comparative statements, such as whether one mechanism is guaranteed to outperform another, would be of interest as well.

Challenge 2: Restricted Domains. For quantitative axioms such as $\varepsilon$-proportionality and $\gamma$-effectiveness, general bounds will often be weak. We may be able to obtain stronger guarantees by considering special classes of representation matrices. For instance, algebraic properties of $\Gamma$ could model salient characteristics of the population relevant to axiomatic analysis: polarized groups would be characterized by a block matrix $\Gamma$, the relative magnitude of $\Gamma$’s trace would quantify the power-seeking agents in the group, the rank of $\Gamma$ would model how correlated agents are to each other, etc.

Challenge 3: Design Characteristics. An intriguing challenge is identifying general characteristics of the selection mechanisms that lead to axiomatic guarantees. In particular, the discussed selection mechanisms differ with respect to multiple dimensions: (i) whether candidates are pre-selected ($m < n$) or anyone can serve on the representative body ($m = n$), (ii) whether the output representative body has a predefined size or not, and (iii) whether each agent has a direct link to some member of the representative body they support ($n$), or some agents are virtually represented (by someone they did not necessarily vote for; $\|f^{M_k}(\Gamma, C)\|_1 < n$). We call these dimensions open-closed, flexible-rigid, and direct-virtual, respectively. Different selection mechanisms are located at different points of the induced 3-dimensional space. We want to understand the impact of these design choices on the axiomatic properties: We envision that mechanisms’ performance may depend on their position in this 3-dimensional space.

Challenge 4: Impossibility Results. Proving that certain axioms are incompatible would allow us to identify currently hidden trade-offs faced by selection mechanisms. In line with axiomatic results from social choice theory [4, 25], we expect to find that domain restrictions can circumvent some impossibility theorems.

Challenge 5: Broadening the Model. The illustrative model discussed in this work is only a first step and can be extended in multiple ways. For instance, extensions to other ballot formats, such as approval or ranked ballots, are natural next steps. Moreover, in addition to analyzing representation-related properties of selection mechanisms, it would also be interesting to study the quality of decisions made by the selected body or its accountability.

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